# MIRROR SYMMETRY AND PROJECTIVE GEOMETRY OF REYE CONGRUENCES I

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ABSTRACT. Studying the mirror symmetry of a Calabi-Yau threefold $X$ of the Reye congruence in $\mathbb{P}^4$, we conjecture that $X$ has a non-trivial Fourier-Mukai partner $Y$. We construct $Y$ as the double cover of a determinantal quintic in $\mathbb{P}^4$ branched over a curve. We also calculate BPS numbers of both $X$ and $Y$ (and also a related Calabi-Yau complete intersection $\tilde{X}_0$) using mirror symmetry.

## 1. Introduction

The set of lines in the $n$-dimensional projective space parametrized by a variety in the Grassmannian $G(2, n+1)$ is called a line congruence. Classical Reye congruences are the line congruences defined by three dimensional linear systems of quadrics in $\mathbb{P}^3$, and have a long history in their study in projective geometry and geometry of quadrics, in particular, in relation to Enriques surfaces [Co, CoD] [1]. As the next generalization, the Reye congruences defined by five quadrics in $\mathbb{P}^4$ [Ol] are of considerable interest since the relevant geometry is given by Calabi-Yau threefolds, where another aspect of mirror symmetry comes into play in addition to the classical ones. In this paper, we study the mirror symmetry of the Reye congruences and find that every Calabi-Yau threefold of the Reye congruence is paired with another Calabi-Yau threefold which arises naturally in the relevant projective geometries of the Reye congruence.

In Section 2, after a brief summary of the Reye congruences in $\mathbb{P}^4$, we define our Calabi-Yau threefold $X$ of the Reye congruences (we shall call Reye congruence $X$ in short hereafter) as a suitable $\mathbb{Z}_2$ quotient of a generic complete intersection $\tilde{X}_0$ of five symmetric $(1, 1)$ divisors in $\mathbb{P}^4 \times \mathbb{P}^4$. Applying the toric method due to Batyrev and Borisov [BB], we first construct a mirror family $\mathcal{X}_0^\vee$ over $\mathbb{P}^2$ and then reduce this family to a diagonal one $\mathcal{X}^\vee$ over $\mathbb{P}^1$ to obtain the mirror of the Reye congruence. We study the period integrals of the both families in details, and find several boundary points called large complex structure limits [Mo]. Among them, we will focus on the two distinct boundary points of maximal unipotent monodromy [Sch] which appear on the reduced family $\mathcal{X}^\vee$ over $\mathbb{P}^1$. Using the mirror symmetry, we calculate the Gromov-Witten invariants at each boundary point, and also the monodromy of the period integrals. Based on these calculations, we identify one of the two boundary points with the mirror of the Reye congruence $X$ as expected, and for the other boundary point we predict a new Calabi-Yau threefold $Y$ that is naturally paired with $X$ (Conjecture 1). The existence of $Y$ is quite similar to the case of a Calabi-Yau threefold given by a transversal linear section of the Grassmannian $G(2, 7)$, where $Y$ appears in the Pfaffian variety $Pf(7)$, the projective

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[1] Recently, the derived category of the classical Reye congruence has been studied in [IKu, Ku3].

dual of $G(2,7)$ [Ro]. In the case of $G(2,7)$ and $Pf(7)$, it has been proved that the two Calabi-Yau threefolds have equivalent derived categories of coherent sheaves; $D(Coh(X)) \cong D(Coh(Y))$ [BCa, Ku1]. We expect that the corresponding property holds also in our case (Conjecture 2).

In Section 3, we shall prove our Conjecture 1 constructing the predicted Calabi-Yau threefold $Y$ in our setting of the Reye congruences. In contrast to that our Reye congruence Calabi-Yau threefold $X$ may be defined by a $\mathbb{Z}_2$-quotient of a suitable complete intersection of five symmetric $(1,1)$ divisors in $\mathbb{P}^4 \times \mathbb{P}^4$, we find that the geometry of $Y$ arises naturally as the $\mathbb{Z}_2$ *covering* of a determinantal quintic in $\mathbb{P}^4$ ramified along a smooth curve of genus 26 and degree 20. In our construction, the projective geometries of the Reye congruences are used efficiently, and there it should be clear that our case is precisely in the line of the case of $G(2,7)$ and $Pf(7)$ studied previously in [BCa, Ku1].

In the final section, we will count some curves in $X$ and $Y$ which are specific in their geometries, and verify the so-called BPS numbers which are read from Gromov-Witten invariants of $X$ and $Y$, respectively.

Main results of this paper are Theorem 3.14 and the BPS numbers of the related Calabi-Yau threefolds. The BPS numbers of $X$ and $Y$ in Tables 1 and 2 in Appendix A, respectively (and also those of $\tilde{X}_0$ in Tables 3,4,5 in Appendix B), are obtained by using mirror symmetry, but should be useful as a testing ground for the recent developments in the study of integral invariants of Calabi-Yau threefolds (see [PT] and references therein).

## 2. **Mirror symmetry of Reye congruences**

**(2-1)** *Reye congruences*: Let $P = |Q_1, Q_2, Q_3, Q_4, Q_5|$ be a 4-dimensional linear system of quadrics in $\mathbb{P}^4$. For a line $l \subset \mathbb{P}^4$, we denote by $W_l(P)$ the linear system of quadrics in $P$ which contain $l$. $P$ is called *regular* if the following conditions are satisfied:

  (i) $P$ is basepoint free.
  (ii) If $l$ is a line in $\mathbb{P}^4$ which is the vertex of some $Q \in P$, then dim $W_l(P) = 1$.

Similar to the classical Reye congruence [Co], for a regular system $P$ of quadrics, we define a Reye congruence $X$ by

$$(2.1) \qquad\qquad X = \{ \, l \subset \mathbb{P}^4 \, | \, \dim W_l(P) = 2 \, \},$$

(cf. a generalized Reye congruence studied in [Ol]). Let $(z, w)$ be the (bi-)homogeneous coordinates of $\mathbb{P}^4 \times \mathbb{P}^4$. The quadrics $Q_i$ define the corresponding bilinear forms:

$$(2.2) \qquad\qquad Q_i(z, w) = {}^t z A_i w, \quad (i = 1, 2, .., 5).$$

We sometimes identify the quadrics $Q_i$ with their associated $5 \times 5$ symmetric matrices $A_i$. The bilinear forms define a complete intersection in $\mathbb{P}^4 \times \mathbb{P}^4$:

$$(2.3) \qquad \tilde{X} : Q_1(z, w) = Q_2(z, w) = \cdots = Q_5(z, w) = 0 .$$

Let $\sigma$ be the involution $\sigma : (z, w) \leftrightarrow (w, z)$. Through the residue formula, the complete intersection $\tilde{X}$ has a holomorphic three form which is invariant under this involution.

**Proposition 2.1.** *(cf. [Ol, Proposition 1.1]) The Reye congruence $X$ is isomorphic to $\tilde{X}/\langle\sigma\rangle$. $\tilde{X}/\langle\sigma\rangle$ is a Calabi-Yau manifold with the Hodge numbers $h^{1,1}(X) = 1, h^{2,1}(X) = 26$.*

*Proof.* We first show that $X \simeq \tilde{X}/\langle\sigma\rangle$. If $(z, w) \in \tilde{X}$, then, by (i), $z \neq w$ and hence $z$ and $w$ span a line $\langle z, w \rangle$. Then the linear system of quadrics in $P$ containing $\langle z, w \rangle$ is $\{Q \in P \mid {}^t z Q z = 0, {}^t w Q w = 0\}$, which has codimension two by (i). Therefore $\langle z, w \rangle \in X$. Conversely, let $l$ be a line in $X$. Then, by the condition (2.1), $P$ induces on $l$ a pencil of 0-dimensional quadrics. Correspondingly, $P$ induces on $l \times l$ a pencil of $(1, 1)$-divisors. Thus there are two base points $(z, w)$ and $(w, z)$ of this pencil, which generate the line $l = \langle z, w \rangle$, and then we have $(z, w) \in \tilde{X}$. Thus $X \simeq \tilde{X}/\langle\sigma\rangle$.

Second we show that, under the condition (i), the condition (ii) implies that $\tilde{X}$ is 3-dimensional and smooth. Indeed, by the Jacobian criterion, $\tilde{X}$ is not 3-dimensional or singular at $(z, w) \in \tilde{X}$ if and only if the line $\langle z, w \rangle$ is contained in the vertex of a quadric in $P$. Since $X \simeq \tilde{X}/\langle\sigma\rangle$, the linear system of quadrics in $P$ containing $\langle z, w \rangle$ is two-dimensional. Thus if the line $\langle z, w \rangle$ is contained in the vertex of a quadric in $P$, this is a contradiction to (ii). (In fact, under (i), the condition (ii) is equivalent to that $\tilde{X}$ is 3-dimensional and smooth.)

Finally, $h^{1,1}(X) = 1$ follows immediately from $h^{1,1}(\tilde{X}) = 2$. $h^{2,1}(X) = 26$ follows from calculating the Euler number $e(\tilde{X}) = 2(h^{1,1}(\tilde{X}) - h^{2,1}(\tilde{X})) = -100$ for the smooth complete intersection $\tilde{X}$, and $e(\tilde{X}) = 2e(X)$, $h^{i,0}(X) = h^{i,0}(\tilde{X}) = 0$ ($i = 1, 2$). $\square$

By using the isomorphism to the quotient of the complete intersection $\tilde{X}$, we obtain the following topological invariants of $X$:

$$\deg(X) = 35 , \ c_2.H = 50 , \ e(X) = -50 ,$$

where $H$ is the ample generator of $Pic(X) \cong \mathbb{Z}$, and $c_2$ is the second Chern class of $X$.

**(2-2)** *Mirror family of the Reye congruence*: The Reye congruence $X$ has a natural covering $\tilde{X} \xrightarrow{2:1} X$ with $\tilde{X}$ a Calabi-Yau complete intersection (2.3). We may have a mirror (family) to our Reye congruence $X$ through the Batyrev-Borisov toric mirror construction [BB] applied to $\tilde{X}$.

$\tilde{X}$ is defined by the zero locus of the five bilinear forms $Q_i(z, w)$ in $\mathbb{P}^4 \times \mathbb{P}^4$. In the toric construction, one considers a complete intersection of generic polynomial equations of bidegree $(1, 1)$ in $\mathbb{P}^4 \times \mathbb{P}^4$ without imposing the invariance under $\sigma$. We denote this generic complete intersection by $\tilde{X}_0$.

Consider $\mathbb{R}^4$ and its dual $\check{\mathbb{R}}^4$ with the pairing $\langle \, , \, \rangle : \check{\mathbb{R}}^4 \times \mathbb{R}^4 \to \mathbb{R}$, and fix the dual bases satisfying $\langle \check{\mathbf{e}}_i, \mathbf{e}_j \rangle = \delta_{ij}$. We denote by $\Delta$ the polytope in $\check{\mathbb{R}}^4$ whose integral points represent the bases of $H^0(\mathbb{P}^4, \mathcal{O}(-K_{\mathbb{P}^4}))$ and defines the toric variety $\mathbb{P}_\Delta =$

$\mathbb{P}^4$. The dual polytope $\Delta^* := \{w \in \mathbb{R}^4 | \langle v, w \rangle \geq -1 (v \in \Delta)\}$ may be written simply by $Conv.(\{\mathbf{e}_1, \cdots, \mathbf{e}_4, -\mathbf{e}_1 - \cdots - \mathbf{e}_4\})$. For the ambient space $\mathbb{P}^4 \times \mathbb{P}^4$, we consider the product $\Delta \times \Delta$ in $\check{\mathbb{R}}^4 \times \check{\mathbb{R}}^4$. Then the vertices of the dual polytope $(\Delta \times \Delta)^*$ are given by $(\mathbf{e}_1, 0), \cdots, (\mathbf{e}_4, 0), (-\mathbf{e}_1 - \cdots - \mathbf{e}_4, 0), (0, \mathbf{e}_1), \cdots, (0, \mathbf{e}_4), (0, -\mathbf{e}_1 - \cdots - \mathbf{e}_4)$. We denote these vertices by $\nu_1, \nu_2, \cdots, \nu_{10}$ in order. One may identify these integral points with the torus invariant divisors $D_{\nu_i} =: D_i$ on $\mathbb{P}_\Delta \times \mathbb{P}_\Delta$.

The integral polytope $\Delta \times \Delta$ is an example of the so-called reflexive polytope, and the generic complete intersection $\tilde{X}_0$ in $\mathbb{P}_{\Delta \times \Delta} = \mathbb{P}_\Delta \times \mathbb{P}_\Delta$ may be defined by the data $N_P = \{\{D_i, D_{i+5}\}\}_{i=1,\cdots,5}$, called a *nef* partition of the anti-canonical class $-K_{\mathbb{P}_{\Delta \times \Delta}} = D_1 + D_2 + \cdots + D_{10}$ (or the vertices of the dual polytope $(\Delta \times \Delta)^*$). Using the data of the nef partition, we have

$$(2.4) \qquad \tilde{X}_0 : f_{\nabla_1} = f_{\nabla_2} = f_{\nabla_3} = f_{\nabla_4} = f_{\nabla_5} = 0 \,,$$

where $f_{\nabla_i}$ is a generic element in $H^0(\mathbb{P}_{\Delta \times \Delta}, \mathcal{O}(D_i + D_{i+5}))$ and $\nabla_i$ is the sub-polytope in $\Delta \times \Delta$ whose integral points represent the bases of the cohomology for $i = 1, \cdots, 5$. By the Lefshetz hyperplane theorem and evaluating the Euler number of $\tilde{X}_0$, one see easily that $\tilde{X}_0$ is a three dimensional Calabi-Yau complete intersection with Hodge numbers $h^{1,1}(\tilde{X}_0) = 2, h^{2,1}(\tilde{X}_0) = 52$.

Define $\nabla = Conv.(\nabla_1, \cdots, \nabla_5)$ in $\check{\mathbb{R}}^4$ and consider its dual polytope $\nabla^*$ in $\mathbb{R}^4$. For the nef partition $N_P$, the polytope $\nabla$ is reflexive and a choice of its triangulation defines a maximally, projective, crepant partial resolution of the associated toric variety $\mathbb{P}_{\nabla^*}$ [BB]. For notational simplicity, we use the same notation $\mathbb{P}_{\nabla^*}$ for such a crepant resolution.

Since the vertices of $\nabla$ represents the toric invariant divisors of the toric variety $\mathbb{P}_{\nabla^*}$, the vertices of $\nabla_i \setminus \{\mathbf{0}\}$ define a nef partition $N_P^\vee$ of $-K_{\mathbb{P}_{\nabla^*}}$ and the corresponding line bundles on $\mathbb{P}_{\nabla^*}$. Then the nef partition $N_P^\vee$ determines sub-polytopes $\Delta_i^*$ ($i = 1, \cdots, 5$) in $(\Delta \times \Delta)^*$, representing the global sections of the line bundles. By duality, it turns out that $\Delta_i^* = Conv(\{0, \nu_i, \nu_{i+5}\})$ and we denote by $f_{\Delta_i^*}$ the generic global sections of the line bundle over $\mathbb{P}_{\nabla^*}$. The data $\Delta_i^*$ provides us the mirror family of Calabi-Yau complete intersections $\tilde{X}_0$:

**Proposition 2.2.** *The generic complete intersection $\tilde{X}_0^\vee$ defined by*

$$(2.5) \qquad f_{\Delta_1^*} = f_{\Delta_2^*} = f_{\Delta_3^*} = f_{\Delta_4^*} = f_{\Delta_5^*} = 0$$

*in the projective toric variety $\mathbb{P}_{\nabla^*}$ is a smooth Calabi-Yau complete intersection with Hodge numbers $h^{1,1}(\tilde{X}_0^\vee) = 52, h^{2,1}(\tilde{X}_0^\vee) = 2$, and defines a (topological) mirror Calabi-Yau manifold to $\tilde{X}_0$.*

The toric variety $\mathbb{P}_{\nabla^*}$ contains the algebraic torus $(\mathbb{C}^*)^8$ as an open dense subset. We denote the coordinates of the tori $(\mathbb{C}^*)^4 \times (\mathbb{C}^*)^4$ by $U_i, V_i$ ($i = 1, \cdots, 4$), respectively. Then the generic global sections (2.5) may be written by

$$(2.6) \qquad f_{\Delta_i^*} = a_i + b_i U_i + c_i V_i \quad (i = 1, \cdots, 5),$$

where $U_5, V_5$ are determined by $U_1 U_2 U_3 U_4 U_5 = 1, V_1 V_2 V_3 V_4 V_5 = 1$, respectively, and $a_i, b_i, c_i$ are generic parameters. With these parameters up to some identifications due to isomorphisms, we have a mirror family $\tilde{\mathcal{X}}_0^\vee$ to $\tilde{X}_0$ after taking the closure of the zero locus in $\mathbb{P}_{\nabla^*}$. In particular, when $b_i = c_i$, we have a diagonal reduction $\tilde{\mathcal{X}}^\vee$ of the generic family $\tilde{\mathcal{X}}_0^\vee$. We note that, for this reduced family, the defining equations are invariant under the involution $\sigma^\vee : U_i \leftrightarrow V_i$ of the tori $(\mathbb{C}^*)^4 \times (\mathbb{C}^*)^4$.

**Lemma 2.3.** *There exists a resolution of the ambient toric variety $\mathbb{P}_{\nabla^*}$ where the involution $\sigma^\vee$ extends from the tori to $\mathbb{P}_{\nabla^*}$.*

*Proof.* The polytope $\nabla_i$ in $\nabla = Conv(\nabla_1, \cdots, \nabla_5)$ is defined by the property that its integral points represent the bases of $H^0(\mathbb{P}_{\Delta \times \Delta}, \mathcal{O}(D_i + D_{i+5}))$. Consider the co-homology $H^0(\mathbb{P}_\Delta, \mathcal{O}(D_i))$ $(1 \leq i \leq 5)$. Through the support function corresponding to the divisor $D_i$ [Oda], it is straightforward to arrive at the polytopes

$$\nabla_5^{(1)} = Conv(\{\mathbf{0}, \check{\mathbf{e}}_1, \check{\mathbf{e}}_2, \check{\mathbf{e}}_3, \check{\mathbf{e}}_4\}) , \ \ \nabla_i^{(1)} = \nabla_5^{(1)} - \check{\mathbf{e}}_i \ (i = 1, \cdots, 4)$$

to represent the cohomology. Since $\mathbb{P}_{\Delta \times \Delta} = \mathbb{P}_\Delta \times \mathbb{P}_\Delta$ and $H^0(\mathbb{P}_{\Delta \times \Delta}, \mathcal{O}(D_i + D_{i+5}))$ is given by the product, $\nabla_i$ is also given by the product $\nabla_i^{(1)} \times \nabla_i^{(1)}$ in $\check{\mathbb{R}}^4 \times \check{\mathbb{R}}^4$.

Now recall that for the projective toric variety we have $\mathbb{P}_{\nabla^*} = \mathbb{P}_{\Sigma(\nabla)}$, where $\Sigma(\nabla)$ is a fan over the faces of the polytope $\nabla$. Precisely a choice of the maximally, projective, crepant partial resolution of $\mathbb{P}_{\nabla^*}$ corresponds to a subdivision of the fan $\Sigma(\nabla)$. We translate the involution $\sigma^\vee$ defined on $(\mathbb{C}^*)^4 \times (\mathbb{C}^*)^4$ as the exchange of the first and the second factor of $\check{\mathbb{R}}^4 \times \check{\mathbb{R}}^4$. Our claim follows if there exists a subdivision of the fan which is invariant under this involution.

It is clear that the involution acts on the set of vertices of $\nabla$ due to the product form $\nabla_i = \nabla_i^{(1)} \times \nabla_i^{(1)}$ and the definition $\nabla = Conv(\nabla_1, \cdots, \nabla_5)$. The set of faces of the polytope $\nabla$ may be determined by a computer program PORTA [PO], for example. Then one can verify that the involution in fact acts on the set of faces. The last property guarantees that there exists the desired subdivision of $\Sigma(\nabla)$.    □

**Proposition 2.4.** *For generic parameters $a_i, b_i = c_i$, the involution on $\tilde{X}^\vee$ is fixed-point free, and the quotient $\tilde{X}^\vee / \langle \sigma^\vee \rangle$ defines a family $\mathcal{X}^\vee$ of a smooth Calabi-Yau manifolds with the Hodge numbers $h^{1,1} = 26$, $h^{2,1} = 1$. We may regard (and will justify) this family as the mirror family of the Reye congruence $X$.*

*Proof.* Consider a generic member $\tilde{X}^\vee$ from the diagonal family $\tilde{\mathcal{X}}^\vee$. On the torus $(\mathbb{C}^*)^8$, using the isomorphisms, one may assume its defining equations are given by

$$(2.7) \quad f_{\Delta_i^*} = a + U_i + V_i \ \ (i = 1, \cdots, 5), \ \ U_1 U_2 U_3 U_4 U_5 = 1, V_1 V_2 V_3 V_4 V_5 = 1 .$$

From this, we see that there is no fixed point in the torus unless $a = -2$. By explicit calculations, one can also verify the same property of $\tilde{X}^\vee$ on each affine chart. The Hodge numbers follow from Proposition 2.2.    □

**Remark 2.5.** The Reye congruence $X$ and its mirror $X^\vee$ are defined as the quotients by the respective involutions. From these constructions, we see that both $X$ and $X^\vee$ have non-trivial fundamental groups (of finite orders).

**(2-3)** *Picard-Fuchs equations ("maximally resonant" GKZ systems)*: Let us denote by $\Omega(a, b, c)$ a holomorphic three form for the mirror family $\tilde{\mathcal{X}}_0^\vee$ defined by (2.6), and consider the period integrals over the three cycles. Since the family is that of complete intersections in a toric variety, the period integral of a torus cycle can be written explicitly [BCo] as

$$(2.8) \qquad \omega(a, b, c) = \frac{1}{(2\pi\sqrt{-1})^8} \int_\gamma \frac{a_1 a_2 a_3 a_4 a_5}{f_{\Delta_1^*} f_{\Delta_2^*} f_{\Delta_3^*} f_{\Delta_4^*} f_{\Delta_5^*}} \prod_{i=1}^4 \frac{dU_i}{U_i} \frac{dV_i}{V_i} ,$$

where $\gamma$ is the torus cycle $|U_i| = |V_i| = 1 (i = 1, \cdots, 4)$. The Picard-Fuchs differential equations satisfied by period integrals of complete intersections in toric varieties

have been studied in general in [HKTY]. In particular, we find our present case $\tilde{X}_0^\vee$ in the examples there (Example 4 in Sect.5).

**Proposition 2.6.** *(1) The period integral $\omega(a, b, c)$ is a function of $x = -\frac{b_1 b_2 b_3 b_4 b_5}{a_1 a_2 a_3 a_4 a_5}$, $y = -\frac{c_1 c_2 c_3 c_4 c_5}{a_1 a_2 a_3 a_4 a_5}$ and satisfies a GKZ hypergeometric system which is given locally by*

$$\{\theta_x^5 - x(\theta_x + \theta_y + 1)^5\}\omega(x, y) = \{\theta_y^5 - y(\theta_x + \theta_y + 1)^5\}\omega(x, y) = 0,$$

*where $\theta_x = x\frac{\partial}{\partial x}, \theta_y = y\frac{\partial}{\partial y}$. Globally, this system is defined over $\mathbb{P}^2$ with $[-x, -y, 1] = [u, v, w] \in \mathbb{P}^2$ and the monodromy is unipotent about the toric divisors $x = 0$ and $y = 0$. Furthermore the system is reducible and its irreducible part determines the all period integrals of the family.*

*(2) In the affine local coordinates $[1, -y_1, -x_1] := [u, v, w] \in \mathbb{P}^2$ and $[-y_2, 1, -x_2] := [u, v, w]$, respectively, the system is represented by the differential equations*

$$\{(\theta_{x_1} - 1)^5 - x_1(\theta_{x_1} + \theta_{y_1})^5\}\tilde{\omega}(x_1, y_1) = \{\theta_{y_1}^5 - y_1(\theta_{x_1} + \theta_{y_1})^5\}\tilde{\omega}(x_1, y_1) = 0,$$

*and the same form of the differential operators for $\tilde{w}(x_2, y_2)$.*

*(3) The monodromy is unipotent also about the toric divisor $w = 0$ ($[u, v, w] \in \mathbb{P}^2$), and there is only one regular solution at every toric boundary point of $\mathbb{P}^2$.*

*Proof.* For the details of the GKZ system, and the derivations of the properties (1), we refer to [HKTY]. The compactification of the parameter space follows from the construction of the secondary fan [GKZ1], which turns out to be isomorphic to the fan of the toric variety $\mathbb{P}^2$. The form of the differential operators in (2) follows in the same way as (1). Then the properties in (3) follow directly from (2). $\qquad \square$

As summarized in the above proposition, the Picard-Fuchs equation of the mirror family $\tilde{\mathcal{X}}_0^\vee$ is defined over $\mathbb{P}^2$ after a natural compactification of the parameter space $a_i, b_i, c_i$. In what follows, we consider our mirror family $\tilde{\mathcal{X}}_0^\vee$ over $\mathbb{P}^2$.

In general, mirror families have boundary divisors about which the monodromies are of finite order. It should be emphasized that our mirror family $\tilde{\mathcal{X}}_0^\vee$ has unipotent monodromies about all the toric boundary divisors. In fact, one can check that all the toric boundary points given by normal crossings of these, are the so-called *Large Complex Structure Limit (LCSL)* points (see [Mo] for a precise definition). Each of the LCSL point will be interpreted from the geometry of $\tilde{X}_0$ in Sect.(3-1).

**Remark 2.7.** 1) The discriminant locus of the GKZ system (1) in Proposition 2.6 consists of the boundary toric divisor $\mathbb{P}^1$'s and the genus 6 nodal curve with 6 nodes in $\mathbb{P}^2$ given by

$$dis_0 := (u + v + w)^5 - 5^4 \, u \, v \, w \, (u + v + w)^2 + 5^5 \, u \, v \, w \, (uv + vw + wu) = 0.$$

Two of the 6 nodes lie on the line $u = v$, and similarly on the lines $v = w$ and $u = w$ for the rest. The two nodes on the line $u = v$ correspond to the conifolds at $x = \alpha_1, \alpha_2$ of the diagonal family $\mathcal{X}^\vee$ (see (2.10) below).

2) The compactification of the parameter space $a_i, b_i, c_i$ is standard due to [GKZ1]. However the minus sign (or more generally the normalization) of the local parameters $x, y$ is not a consequence of this compactification. This is the *right* normalization observed widely in [HKTY] for the mirror map and also for the right predictions of Gromov-Witten invariants of complete intersection Calabi-Yau manifolds.

**(2-4)** *Picard-Fuchs equations of $\mathcal{X}^\vee$*: We may consider the symmetric reduction $a_i, b_i = c_i$ of the family simply by setting $x = y$. Then the Picard-Fuchs differential equation satisfied by the period integrals of the $\sigma^\vee$-quotient family $\mathcal{X}^\vee$ follows from Proposition 2.6 as follows:

**Proposition 2.8.** *The mirror family $\mathcal{X}^\vee$ of the Reye congruence $X$ is defined over the diagonal $\mathbb{P}^1$ of $x = y$. The period integrals of this family satisfies the Picard-Fuchs equation $\mathcal{D}\omega(x) = 0$ with $\theta_x = x\frac{d}{dx}$ and*

(2.9)
$$
\begin{aligned}
\mathcal{D} =& 49\,\theta_x^4 - 7x(155\theta_x^4 + 286\theta_x^3 + 234\theta_x^2 + 91\theta_x + 14) \\
& - x^2(16105\theta_x^4 + 680044\theta_x^3 + 102261\theta_x^2 + 66094\theta_x + 15736) \\
& + 8x^2(2625\theta_x^4 + 8589\theta_x^3 + 9071\theta_x^2 + 3759\theta_x + 476) \\
& - 16x^4(465\theta_x^4 + 1266\theta_x^3 + 1439\theta_x^2 + 806\theta_x + 184) + 512x^5(\theta_x + 1)^4.
\end{aligned}
$$

The above differential equation appeared first in [BaS] for the mirror family of $\tilde{X}$ in the early stage of mirror symmetry. We should note, however, that we are considering the same equation for the mirror family of our Reye congruence $X$, i.e., with the involution $\sigma^\vee$. This difference of interpretation becomes crucial when calculating higher-genus Gromov-Witten invariants in Sect.(2-6.2) and (2-6.3).

From the form of the Picard-Fuchs equation (2.9), we observe that $x = 0$ is a singular point with maximally unipotent monodromy, which is equivalent to the LCSL point for one dimensional families [Mo]. We also observe that $x = \infty$ has the same property. Our interest hereafter will be to reveal a Calabi-Yau geometry which comes naturally with our Reye congruence and appears near this latter boundary point. Similar phenomena has been observed first by Rødland for a Calabi-Yau complete intersection in a Grassmannian and a certain Pfaffian variety extracting genus zero Gromov-Witten invariants [Ro]. In case of the Grassmannian and Pfaffian Calabi-Yau manifolds, higher genus Gromov-Witten invariants have been calculated recently from the Picard-Fuchs equation [HK]. After a summary of the properties of our Picard-Fuchs equation (2.9), we will determine the higher genus Gromov-Witten invariants which come out from the two different LCSL points.

**(2-5)** *Monodromy calculations*: The Picard-Fuchs equation is a differential equation of Fuchs type with its all singularities being regular. The entire picture can be grasped by the following Riemann's $\mathcal{P}$-symbol:

(2.10)
$$
\left\{
\begin{array}{c|cccccc}
x & 0 & 1/32 & \alpha_1 & \alpha_2 & 7/4 & \infty \\
\hline
\rho_1 & 0 & 0 & 0 & 0 & 0 & 1 \\
\rho_2 & 0 & 1 & 1 & 1 & 1 & 1 \\
\rho_3 & 0 & 1 & 1 & 1 & 3 & 1 \\
\rho_4 & 0 & 2 & 2 & 2 & 4 & 1
\end{array}
\right\},
$$

where $\alpha_k$ are the roots of the equation $1 + 11x - x^2 = 0$. The singularities at $1/32, \alpha_1 = (11 - 5\sqrt{5})/2, \alpha_2 = (11 + 5\sqrt{5})/2$ are called *conifold* in general (although $1/32$ is slightly different from the latter two, see Proposition 2.9 below). Whereas at the point $x = 7/4$, called apparent singularity, there is no monodromy around the point.

**Proposition 2.9.** *The fiber of the mirror family $\mathcal{X}^\vee$ over $\mathbb{P}^1$ has ordinary double points over $x = \alpha_1, \alpha_2$, and a $\mathbb{Z}_2$ quotient of the ordinary double point over $x = 1/32$.*

*Proof.* From the Picard-Fuchs equation, we see that the singularities at the boundaries are maximally unipotent. For others, we study the Jacobian ideal of (2.7): $a + U_i + V_i = 0$ $(i = 1, \cdots, 5), U_1 \cdots U_5 = V_1 \cdots V_5 = 1$. By computing a Gröbner basis with a suitable term order, we see that it consists of equations $V_1 = V_2 = V_3 = V_4 = V_5$, $a + U_i + V_i = 0$ $(i = 1, \cdots, 5)$, $V_5^5 - 1 = 0$ and five other higher order equations of $V_5$ and $a$, which result in the discriminant when $V_5$ is eliminated. The defining equations are invariant under $(U_i, V_i, a) \to (\lambda U_i, \lambda V_i, \lambda a)$ for $\lambda^5 = 1$, and so is the Jacobian ideal. With this $\mathbb{Z}_5$ action, one may summarize all the zeros of the Jacobian ideal into the following orbits:

$$\mathbb{Z}_5 \cdot (\text{-1-}a, \text{-1-}a, \text{-1-}a, \text{-1-}a, \text{-1-}a, 1, 1, 1, 1, 1, a)$$

for each solution of $(1 + a)^5 = -1$. It is easy to see that these zeros are ordinary double points in $X_a^\vee$. Hence each orbit represents one ordinary double point in (the isomorphism class of) $X_x^\vee$ parametrized by $x = -\frac{1}{a^5}$. Thus we have five ordinary double points corresponding to each solution of $(1+a)^5 = -1$. We see that the five solutions are mapped to $x = \alpha_1, \alpha_2, 1/32$, with two ordinary double points being identified under the involution $\sigma^\vee$ for each $\alpha_1, \alpha_2$ and making a $\mathbb{Z}_2$ quotient over $x = 1/32$.                                                                                           $\square$

Since the Picard-Fuchs equation (2.9) is not of hypergeometric type, there is little hope to describe its integral basis for the solutions analytically. However, we may put several ansatz and requirements coming from mirror symmetry. Combined these with numerical calculations which are available for example in Maple [Ma], we can arrive at an integral basis.

(2-5.1) Near $x = 0$, there exists a unique regular solution up to normalization. We fix the solution by a closed formula that come from the GKZ hypergeometric series:

$$w_0(x) = \sum_{n,m \geq 0} \frac{((n+m)!)^5}{(n!)^5(m!)^5} x^n y^n|_{x=y} = 1 + 2x + 34x^2 + \cdots .$$

(2-5.2) All other solutions contain logarithmic singularities, which may be explained by the Frobenius method applied to GKZ system [HKTY]. We fix these solutions by requiring the following form:

$$w_1(x) = w_0(x) \log x + w_1^{reg}(x),$$
$$w_2(x) = -w_0(x)(\log x)^2 + 2w_1(x) \log x + w_2^{reg}(x),$$
$$w_3(x) = w_0(x)(\log x)^3 - 3w_1(x)(\log x)^2 + 3w_2(x) \log x + w_3^{reg}(x),$$

where $w_k^{reg}(x)$ represents the regular part of the solution $w_k(x)$, and we require that the series expansion of $w_k^{reg}(x)/w_0(x)$ does not contain a constant term for each $k = 1, 2, 3$.

(2-5.3) Define the mirror map, $x = x(q)$, by inverting the series

$$q = e^{\frac{w_1(x)}{w_0(x)}} = x(1 + 5x + 90x^2 + \cdots).$$

(2-5.4) Local solutions around $x = \infty$ may be arranged in the same way above. Setting $z = \frac{1}{x}$, we normalize the regular solution

$$\tilde{w}_0(z) = 2 \sum_{n \geq 1} {}_5F_4(n^5, 1^4; -1) z^n = z + \frac{1}{2}z^2 + \frac{227}{128}z^3 + \frac{4849}{512}z^4 \cdots ,$$

where $_5F_4(n^5, 1^4; x)$ is the generalized hypergeometric series. For the other solutions, similarly to (2-5.2), we set the following ansatz:

$$\tilde{w}_1(z) = \tilde{w}_0(z) \log cz + \tilde{w}_1^{reg}(z),$$
$$\tilde{w}_2(z) = -\tilde{w}_0(z)(\log cz)^2 + 2\tilde{w}_1(z) \log cz + \tilde{w}_2^{reg}(z),$$
$$\tilde{w}_3(z) = \tilde{w}_0(z)(\log cz)^3 - 3\tilde{w}_1(z)(\log cz)^2 + 3\tilde{w}_2(z) \log cz + \tilde{w}_3^{reg}(z),$$

with some constant $c$, and require that the series expansion of $\tilde{w}_k^{reg}(z)/\tilde{w}_0(z)$ does not contain a constant term for each $k$.

(2-5.5) Denote by $\Omega_x$ a holomorphic three form on the mirror Calabi-Yau threefold over a point $x \in \mathbb{P}^1$ of the family $\mathcal{X}^\vee$. Fix a symplectic basis $\{A_0, A_1, B^1, B^0\}$ of $H^3(X_{x_0}^\vee, \mathbb{Z})$ with its symplectic form $\mathsf{S} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$, and define the Period integrals; $^t\Pi(x) = (\int_{A_0} \Omega_x, \int_{A_1} \Omega_x, \int_{B^1} \Omega_x, \int_{B^0} \Omega_x)$. Using the mirror symmetry which arises near LCSL [CdOGP], we make the following ansatz for the period integrals in terms of the local solutions:

$$\Pi(x) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \beta & a & \kappa/2 & 0 \\ \gamma & \beta & 0 & -\kappa/6 \end{pmatrix} \begin{pmatrix} n_0 w_0(x) \\ n_1 w_1(x) \\ n_2 w_2(x) \\ n_3 w_3(x) \end{pmatrix} , \quad \tilde{\Pi}(z) = N_z \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \tilde{\beta} & \tilde{a} & \tilde{\kappa}/2 & 0 \\ \tilde{\gamma} & \tilde{\beta} & 0 & -\tilde{\kappa}/6 \end{pmatrix} \begin{pmatrix} n_0 \tilde{w}_0(z) \\ n_1 \tilde{w}_1(z) \\ n_2 \tilde{w}_2(z) \\ n_3 \tilde{w}_3(z) \end{pmatrix} ,$$

where $\kappa = \deg(X), \beta = -\frac{c_2.H}{24}, \gamma = -\frac{\zeta(3)}{(2\pi i)^3} e(X)$ with the topological invariants of the Reye congruence, i.e., $\deg(X) = 35, c_2.H = 50, e(X) = -50$ and $a$ is an unknown parameter with no geometric interpretation, also we set $n_k = \frac{1}{(2\pi i)^k}$. $\tilde{\Pi}(z)$ is supposed to be a symplectic transform of $\Pi(x)$ and for the parameters $\tilde{\kappa}, \tilde{\beta}, \tilde{\gamma}$, similar geometric interpretations are expected for the solutions about $z = 0$.

(2-5.6) With a choice of the holomorphic three form, the Griffith-Yukawa coupling is defined by $C_{xxx} := -\int_{X_x^\vee} \Omega_x \wedge \frac{d^3}{dx^3} \Omega_x$. By the fact that period integrals satisfy the Picard-Fuchs equation (2.9), we can determine it up to some constant [CdOGP],

$$C_{xxx} = -{}^t\Pi(x) \mathsf{S} \left( \frac{d^3}{dx^3} \Pi(x) \right) = \frac{K(35 - 20x)}{x^3(1 - 32x)(1 + 11x - x^2)},$$

where the constant $K$ will be fixed to 1 later to have a right normalization of the quantum cohomology ring of $X$. Also we have the following relations expressing the Griffiths transversality:

$$^t\Pi(x) \mathsf{S} \left( \frac{d}{dx} \Pi(x) \right) = {}^t\Pi(x) \mathsf{S} \left( \frac{d^2}{dx^2} \Pi(x) \right) = 0.$$

(2-5.7) The relations in (2-5.6) provide conditions for the ansatz of period integrals in (2-5.5). In addition to these, we should have another constraint;

$$^t\Pi(x) \mathsf{S} \left( \frac{d^3}{dx^3} \Pi(x) \right)(-x^2)^3 = {}^t\tilde{\Pi}(z) \mathsf{S} \left( \frac{d^3}{dz^3} \tilde{\Pi}(z) \right),$$

which expresses the relation $C_{xxx} (\frac{dx}{dx})^3 = C_{zzz}$.

We have obtained the following results once passing to a numerical calculations (see [EnS] for example).

**Proposition 2.10.** *(1) There exists a solution for the ansatz (2-5.5) of integral, symplectic basis of the solutions of Picard-Fuchs equation (2.9) with*

$$\tilde{\kappa} = 10 \ , \ \tilde{\beta} = -\frac{40}{24} \ , \ \tilde{\gamma} = -\frac{\zeta(3)}{(2\pi i)^3}(-50) \ , \ c = \frac{1}{2^5} \ , \ a = -\frac{1}{2} \ , \ \tilde{a} = 0 \ , \ N_z = \frac{1}{4}.$$

(2) *When the analytic continuation is performed along a path in the upper half plane, the two local solutions are related by a symplectic matrix* $\Pi(x) = S_{xz}\tilde{\Pi}(x)$ *with* $S_{xz} = \begin{pmatrix} -2 & 5 & -1 & 4 \\ 0 & 2 & 0 & 1 \\ 5 & -1 & 3 & -3 \\ 0 & -5 & 0 & -3 \end{pmatrix}$.

(3) *Monodromy matrices at each singular point have the forms given in Table 1.*

| | $x = \alpha_1$ | $0$ | $1/32$ | $\alpha_2$ | $\infty$ |
|---|---|---|---|---|---|
| $\Pi(x)$ | $\begin{pmatrix} 11 & -14 & 2 & 4 \\ 5 & -6 & 1 & 2 \\ 35 & -49 & 8 & 14 \\ -25 & 35 & -5 & -9 \end{pmatrix}$ | $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 17 & 35 & 1 & 0 \\ -10 & -18 & -1 & 1 \end{pmatrix}$ | $\begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ | $\begin{pmatrix} 1 & 10 & 0 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & -25 & 1 & -10 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ | $\begin{pmatrix} 41 & -34 & 12 & 30 \\ 4 & -6 & 1 & 2 \\ -72 & 51 & -22 & -56 \\ -15 & 18 & -4 & -9 \end{pmatrix}$ |
| $\tilde{\Pi}(z)$ | $\begin{pmatrix} -19 & 0 & -8 & 16 \\ 10 & 1 & 4 & -8 \\ 0 & 0 & 1 & 0 \\ -25 & 0 & -10 & 21 \end{pmatrix}$ | $\begin{pmatrix} -19 & 170 & -8 & 105 \\ 9 & -69 & 4 & -43 \\ 5 & -110 & 1 & -65 \\ -20 & 145 & -9 & 91 \end{pmatrix}$ | $\begin{pmatrix} 1 & 30 & 0 & 18 \\ 0 & 1 & 0 & 0 \\ 0 & -50 & 1 & -30 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ | $\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ | $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 5 & 10 & 1 & 0 \\ -5 & -5 & -1 & 1 \end{pmatrix}$ |

**Table 1.** Monodromy matrices.

Fitting the period integral $\tilde{\Pi}(z)$ into the form (2-5.5) which admit mirror interpretation, we come to the following conjecture for the mirror geometry that emerges about the second LCSL point at $x = \infty$:

**Conjecture 1.** *There exists a smooth projective Calabi-Yau threefold $Y$ with its topological invariants*

$$deg(Y) = 10 \ , \ \ c_2.H = 40 \ , \ \ e(Y) = -50 \ , \ \ h^{1,1}(Y) = 1, \ \ h^{2,1}(Y) = 26,$$

*where $H$ is the ample generator of the Picard group $Pic(Y) \cong \mathbb{Z}$.*

The existence of $Y$ may be expected from the table of 'Calabi-Yau differential equations' of fourth order in [EnS]. Here we have arrived at Conjecture 1 starting from the Reye congruence $X$. One should note that $Y$ is not birational to $X$ since $\rho(X) = \rho(Y) = 1$ and $deg(X) \neq deg(Y)$. The appearring relation between the two is reminiscent of the example of Calabi-Yau threefolds related to Grassmannian and Pfaffian [Ro][BCa], which has been understood in the proposal 'homological projective duality' [Ku1]. In this context, we naturally come to:

**Conjecture 2.** *The Reye congruence $X$ and $Y$ have equivalent derived categories of coherent sheaves; $D(Coh(X)) \cong D(Coh(Y))$.*

We will prove Conjecture 1 in the following section while we have to leave Conjecture 2 for future investigations. Here we remark that analogous conjectures may be stated for odd-dimensional Reye congruences in general, since one can observe that Picard-Fuchs equations have similar properties in odd-dimensions.

**(2-6)** *Gromov-Witten invariants*: One of the important applications of the mirror symmetry is predicting Gromov-Witten invariants of Calabi-Yau manifolds [CdOGP], [BCOV]. Combined with Conjecture 1, we can extract Gromov-Witten invariants for $X$ and $Y$ from the LCSL point at $x = 0$ and $z = 0$, respectively.
(2-6.1) The genus 0 Gromov-Witten invariants of the Reye congruence $X$ can be read from the $q$ expansions ($q := e^t$) of the Griffiths-Yukawa coupling,

$$\frac{1}{w_0(x)^2}C_{xxx}\left(\frac{dx}{dt}\right)^3 = 35 + \sum_{d \geq 1} d^3 N_0^X(d) q^d \ ,$$

with $K = 1$ in $C_{xxx}$ to have $35 = \deg(X)$ at the constant term. For the conjectural geometry $Y$, we define the mirror map $z = z(\tilde{q})$ by inverting the series $\tilde{q} = e^{\frac{\tilde{w}_1(z)}{\tilde{w}_0(z)}} = cz(z + \frac{35z^2}{16} + \frac{10395z^3}{1024} + \cdots)$, with $c = \frac{1}{2^5}$. Then genus 0 Gromov-Witten invariants of $Y$ are read from

$$\frac{1}{\tilde{w}_0(z)^2} C_{zzz} \left(\frac{dz}{d\tilde{t}}\right)^3 = 10 + \sum_{d \geq 1} d^3 \, N_0^Y(d) \, \tilde{q}^d \; ,$$

with $K = 1$ in $C_{zzz} = C_{xxx}(\frac{dx}{dz})^3$ as fixed above. The genus one Gromov-Witten invariants follow similarly by using the BCOV formula in [BCOV] with the topological invariants of $X$ and those given in Conjecture 1 for $Y$.

(2-6.2) Extracting higher genus ($g \geq 2$) Gromov-Witten invariants from period integrals becomes more involved than above. However calculations are essentially the same as those formulated in the Grassmannian and Pfaffian ([HK] and references therein). We simply present the resulting BPS numbers, which are determined from Gromov-Witten invariants (see Appendix A).

(2-6.3) We can also determine Gromov-Witten invariants for the covering $\tilde{X}$ (or $\tilde{X}_0$). Higher genus calculations in this case are more complicated and time consuming. The details will be reported elsewhere, and we simply list the resulting BPS numbers $g = 0, 1, 2$ in Appendix B.

## 3. Projective duality and the double covering

**(3-1)** *Mukai dual of $\tilde{X}_0$:* Here we interpret the three LCSL points observed in the mirror family $\tilde{\mathcal{X}}_0^\vee$ to the Calabi-Yau manifold $\tilde{X}_0$.

Recall that $\tilde{X}_0$ is defined as a complete intersection of five generic global sections of $H^0(\mathbb{P}^4 \times \mathbb{P}^4, \mathcal{O}(1,1))$. Explicitly one may write the defining equations as

$${}^t z \mathsf{A}_1 w = {}^t z \mathsf{A}_2 w = {}^t z \mathsf{A}_3 w = {}^t z \mathsf{A}_4 w = {}^t z \mathsf{A}_5 w = 0,$$

where $\mathsf{A}_i$ are general $5 \times 5$ **C**-matrices, which generalize the corresponding symmetric matrices $A_i$ in Sect.(2-1). Consider the Segre embedding $\mathbb{P}^4 \times \mathbb{P}^4 \hookrightarrow \mathbb{P}^{24} = \mathrm{Proj}\mathbf{C}[x_{ij}|1 \leq i, j \leq 5]$ by $x_{ij} = z_i w_j$. Then the global sections of the $(1,1)$ divisor extend to linear forms $H_i \, (i = 1, \cdots, 5)$ on $\mathbb{P}^{24}$, and we have

$$\tilde{X}_0 = (\mathbb{P}^4 \times \mathbb{P}^4) \cap H_1 \cap \cdots \cap H_5 \; \subset \mathbb{P}^{24} \; .$$

Denote by $\check{\mathbb{P}}^{24} := (\mathbb{P}^{24})^*$ the dual projective space to $\mathbb{P}^{24}$ and by $(\mathbb{P}^4 \times \mathbb{P}^4)^*$ the dual variety in $\check{\mathbb{P}}^{24}$ to the Segre embedding in $\mathbb{P}^{24}$. According to Mukai, we define a modified dual of $\tilde{X}_0$ to be

$$\tilde{X}_0^\sharp := (\mathbb{P}^4 \times \mathbb{P}^4)^* \cap \langle h_1, \cdots, h_5 \rangle \subset \check{\mathbb{P}}^{24} \; ,$$

where $\langle h_1, \cdots, h_5 \rangle$ represents the projective space spanned by the dual points $h_i$ to $H_i$. We call this the *Mukai dual* to $\tilde{X}_0$. The following is a classical result:

**Lemma 3.1.** $(\mathbb{P}^4 \times \mathbb{P}^4)^* = \{[\check{c}_{ij}] \in \check{\mathbb{P}}^{24} \,|\, \det(\check{c}_{ij}) = 0\}$.

*Proof.* Consider a hyperplane $\mathcal{H} := \sum_{ij} \check{c}_{ij} x_{ij} = 0$ in $\mathbb{P}^{24}$. Denote by $D$ the restriction of $\mathcal{H}$ to $\mathbb{P}^4 \times \mathbb{P}^4 \subset \mathbb{P}^{24}$. The dual variety consists of the points $[\check{c}_{ij}]$ in $\check{\mathbb{P}}^{24}$ for which $\mathcal{H}$ is tangent to $\mathbb{P}^4 \times \mathbb{P}^4$, i.e., $D$ is singular in $\mathbb{P}^4 \times \mathbb{P}^4$. Using the equation $\sum \check{c}_{ij} z_i w_j = 0$ of $D$, and the Jacobian criterion, it is straightforward to see that the condition $\det(\check{c}_{ij}) = 0$ is equivalent for $D$ to be singular. $\qquad\square$

It is well-known that $\mathrm{Sing}\,(\mathbb{P}^4 \times \mathbb{P}^4)^*$ is the locus of $5 \times 5$ matrices of rank less than or equal to three in $\check{\mathbb{P}}^{24}$.

**Lemma 3.2.** $\deg \mathrm{Sing}\,(\mathbb{P}^4 \times \mathbb{P}^4)^* = 50$. *Let* $\mathsf{A}$ *be a* $5 \times 5$ *matrix of rank three. Then, analytically locally near* $[\mathsf{A}]$, $(\mathbb{P}^4 \times \mathbb{P}^4)^*$ *is isomorphic to the product of the* 3-*dimensional ordinary double point and* $\mathbb{C}^{20}$. *In particular,* $\mathrm{codim}\,\mathrm{Sing}\,(\mathbb{P}^4 \times \mathbb{P}^4)^* = 3$.

*Proof.* The first statement is a special case of [HTu, Proposition 12]. The second statement can be proved in a similar way to [Ty, Chap. 2, §3, Lemma 2.3]. We include a proof for the readers' convenience. We have only to determine the tangent cone of $(\mathbb{P}^4 \times \mathbb{P}^4)^*$ at $[\mathsf{A}]$. We may assume that ${}^t z \mathsf{A} w = z_1 w_1 + z_2 w_2 + z_3 w_3$. Let $\mathsf{B} = (b_{ij})$ be a $5 \times 5$ matrix. The line $\mathsf{A} + t\mathsf{B}$ in $\check{\mathbb{P}}^{24}$ is contained in the tangent cone of $(\mathbb{P}^4 \times \mathbb{P}^4)^*$ at $[\mathsf{A}]$ if and only if the degree two term of $\det(\mathsf{A} + t\mathsf{B})$ vanishes, equivalently, $\det \begin{vmatrix} b_{44} & b_{45} \\ b_{54} & b_{55} \end{vmatrix} = 0$. This implies that the tangent cone of $(\mathbb{P}^4 \times \mathbb{P}^4)^*$ at $[\mathsf{A}]$ is the cone over the smooth quadric surface $\{b_{44}b_{55} - b_{45}b_{54} = 0\}$ in $\mathbb{P}^3$ with vertex $\mathbb{P}^{20}$. $\qquad\square$

**Proposition 3.3.** *The dual variety* $\tilde{X}_0^\sharp$ *is a determinantal quintic:* $\det(\sum_{i=1}^5 y_i \mathsf{A}_i) = 0$ *in* $\check{\mathbb{P}}_h^4 := \langle h_1, \cdots, h_5 \rangle$. $\mathrm{Sing}\,\tilde{X}_0^\sharp$ *consists of* 50 *ordinary double points.*

*Proof.* Since the Segre embedding is defined by $x_{ij} = z_i w_j$, the linear forms may be written $H_k = {}^t z \mathsf{A}_k w = \sum_{i,j} \mathsf{a}_{ij}^k x_{ij}$ with $\mathsf{A}_k = (\mathsf{a}_{ij}^k)$. Then the dual points to the hyperplanes are given by $h_k = [\mathsf{a}_{ij}^k] \in \check{\mathbb{P}}^{24}$. Therefore the elements of $\check{\mathbb{P}}_h^4 := \langle h_1, \cdots, h_5 \rangle$ are written by $\sum_k y_k [\mathsf{a}_{ij}^k]$. Then the first statement follows from the Lemma 3.1. For a general choice of $\mathsf{A}_i$, $\tilde{X}_0^\sharp$ is a general linear section of $(\mathbb{P}^4 \times \mathbb{P}^4)^*$. Thus, by Lemma 3.2, the last statement follows. $\qquad\square$

**(3-2)** *The three LCSL points in Proposition 2.6*: Define matrices $\mathsf{B}_i, \mathsf{C}_i$ by the following relations:

$$\sum y_i \mathsf{A}_i = {}^t(\mathsf{B}_1 y \ \mathsf{B}_2 y \ \mathsf{B}_3 y \ \mathsf{B}_4 y \ \mathsf{B}_5 y) = (\mathsf{C}_1 y \ \mathsf{C}_2 y \ \mathsf{C}_3 y \ \mathsf{C}_4 y \ \mathsf{C}_5 y),$$

where $y = {}^t(y_1 \ y_2 \ \cdots y_5)$. The $5 \times 5$ matrices $\mathsf{B}_i, \mathsf{C}_i$ and $\mathsf{A}_i$ are different in general. Define Calabi-Yau complete intersections $\tilde{X}_1, \tilde{X}_2$ in $\mathbb{P}^4 \times \mathbb{P}^4$ by

$$\tilde{X}_1 = \{(y, w) \mid (\sum y_i \mathsf{A}_i) w = 0\} = \{(y, w) \mid {}^t y \, {}^t \mathsf{B}_i w = 0 \, (1 \le i \le 5)\},$$

$$\tilde{X}_2 = \{(z, y) \mid {}^t z(\sum y_i \mathsf{A}_i) = 0\} = \{(z, y) \mid {}^t z \mathsf{C}_i y = 0 \, (1 \le i \le 5)\}.$$

We also set

$$Z_1 = \{w \in \mathbb{P}^4 \mid \det(\mathsf{A}_1 w \cdots \mathsf{A}_5 w) = 0\}, \ Z_2 = \{z \in \mathbb{P}^4 \mid \det({}^t z \mathsf{A}_1 \cdots {}^t z \mathsf{A}_5) = 0\}.$$

Then we see that there exists the following diagram:

(3.1)

$$
\begin{array}{ccccc}
\tilde{X}_1 & \dashleftarrow & \tilde{X}_0 & \dashrightarrow & \tilde{X}_2 \\
\swarrow & & \swarrow \ \ \searrow & & \searrow \\
\tilde{X}_0^\sharp & \searrow & Z_1 \qquad Z_2 & & \tilde{X}_0^\sharp
\end{array}
$$

Both $\tilde{X}_1$ and $\tilde{X}_2$ are smooth for generic choices of $\mathsf{A}_i$. It is easy to see that $\det \sum y_i \mathsf{A}_i = 0$ for $(y, w) \in \tilde{X}_1$. Hence by the correspondence $(y, w) \mapsto \sum y_i \mathsf{A}_i$,

we have a map $\tilde{X}_1 \to \tilde{X}_0^\sharp$. Since $rk \sum y_i \mathbf{A}_i = 4$ for a smooth point $y$ of $\tilde{X}_0^\sharp$, the map $\tilde{X}_1 \to \tilde{X}_0^\sharp$ is bijective except the 50 singular points. Therefore $\tilde{X}_1 \to \tilde{X}_0^\sharp$ is a resolution, which is crepant since both the canonical bundles of $\tilde{X}_1$ and $\tilde{X}_0^\sharp$ are trivial. Since $\tilde{X}_1 \to \tilde{X}_0^\sharp$ is crepant, it is a small resolution of 50 ordinary double points. Similarly, we can define a map $\tilde{X}_2 \to \tilde{X}_0^\sharp$ by the correspondence $(z, y) \mapsto \sum y_i \mathbf{A}_i$, which is another crepant resolution of $\tilde{X}_0^\sharp$.

By the natural projections, we have maps $\tilde{X}_i \to Z_i$ and $\tilde{X}_0 \to Z_i$ $(i = 1, 2)$, all of which are crepant resolutions. Since $Z_1$ and $Z_2$ are also generic determinantal quintics, they have 50 ordinary double points respectively by Lemma 3.2, and then $\tilde{X}_i \to Z_i$ and $\tilde{X}_0 \to Z_i$ $(i = 1, 2)$ are small resolutions.

In general, $\tilde{X}_i$ and $\tilde{X}_0$ are not isomorphic over $Z_i$ $(i = 1, 2)$. Indeed, if $\tilde{X}_1$ and $\tilde{X}_0$ were isomorphic over $Z_1$ for example, then $\tilde{X}_0^\sharp$ and $Z_2$ would be isomorphic but this is impossible for a general choice of $\mathbf{A}_i$. Hence $\tilde{X}_0 \dashrightarrow \tilde{X}_i$ $(i = 1, 2)$ are the Atiyah flops since all the Picard numbers of $\tilde{X}_i$ and $\tilde{X}_0$ are two. Thus all of $\tilde{X}_i (i = 0, 1, 2)$ are smooth Calabi-Yau 3-folds which are birational to each other. In particular, they have the equivalent derived categories due to [Br].

Note that $\tilde{X}_0, \tilde{X}_1$ and $\tilde{X}_2$ are all complete intersections of five $(1, 1)$-divisors in $\mathbb{P}^4 \times \mathbb{P}^4$, and thus in the same deformation family. Then, by the Batyrev-Borisov mirror construction, we see that they share the same mirror family $\tilde{\mathcal{X}}_0^\vee$ over $\mathbb{P}^2$, where we have found three LCSL points (Proposition 2.6). One may naturally consider that the geometry of $\tilde{X}_i$ $(i = 0, 1, 2)$ appears near each LCSL point under the mirror symmetry. This is reminiscent of the topology change studied in [AGM], however, one should note that our flops do not come from those of the ambient space.

**(3-3)** *$Sym^2 \mathbb{P}^4$ and projective duality*: Consider the Chow variety $\mathrm{Chow}^2 \mathbb{P}^4$ of 0-cycles of degree 2 in $\mathbb{P}^4 = \mathbb{P}(\mathbb{C}^5)$. A Chow variety, in general, may be embedded into a projective variety which is defined by the so-called Chow form. Let $x + y$ be a 0-cycle of degree 2 in $\mathbb{P}^4$. The Chow form in this case is given by the product of two linear forms $(x \cdot \xi)(y \cdot \xi)$ with $\xi \in \check{\mathbb{C}}^5$. Then the embedding $\mathrm{Chow}^2 \mathbb{P}^4 \hookrightarrow \mathbb{P}(Sym^2 \mathbb{C}^5)$ is defined by $x + y \mapsto [p_{ij}(x, y)]$ with $(x \cdot \xi)(y \cdot \xi) = \sum_{i \geq j} p_{ij}(x, y) \xi_i \xi_j$. Since $p_{ij}(x, y) = \frac{1}{2}(x_i y_j + x_j y_i)$ $(1 \leq i, j \leq 5)$, we see that the embedding is given by the linear system of symmetric $(1, 1)$-divisors on $\mathbb{P}^4 \times \mathbb{P}^4$. Since the global sections of symmetric $(1, 1)$-divisors generate symmetric polynomials in $\mathbf{C}[x_i, y_j]$, we see the isomorphism $\mathrm{Chow}^2 \mathbb{P}^4 \cong Sym^2 \mathbb{P}^4$ as an algebraic variety [GKZ2].

Our Reye congruence $X$ (2.1) is defined as a subvariety in the Grassmannian $G(2, 5)$. On the other hand, the isomorphic quotient $\tilde{X}/\langle \sigma \rangle$ may be regarded as a subvariety in $Sym^2 \mathbb{P}^4$. We remark that these two are connected by the natural diagram: $G(2, 5) \leftarrow \mathrm{Hilb}^2 \mathbb{P}^4 \to \mathrm{Chow}^2 \mathbb{P}^4 \cong Sym^2 \mathbb{P}^4$ with the standard morphisms.

Let $\Sigma := Sym^2 \mathbb{P}^4$, and $\Sigma_0 := \mathrm{Sing}\,\Sigma$ be the singular locus of $\Sigma$. $\Sigma_0$ is the second Veronese variety $v_2(\mathbb{P}^4)$, namely, $\mathbb{P}^4$ embedded in $\mathbb{P}^{14}$ by the linear system of quadrics.

**Proposition 3.4.** *$\Sigma$ is the secant variety of $\Sigma_0$, i.e., $\Sigma = \overline{\cup\{\langle p, q \rangle \mid p, q \in \Sigma_0, p \neq q\}}$, where $\langle p, q \rangle$ is the line through $p$ and $q$.*

*Proof.* Note the identity for the Chow embedding

$$p_{ij}(x + y, x + y) - p_{ij}(x - y, x - y) = 4p_{ij}(x, y).$$

This implies the correspondence between the point $(x, y) \in \mathrm{Sym}^2\mathbb{P}^4 \cong \Sigma$ $(x \neq y)$ and the line $\langle x+y, x-y \rangle \in \Sigma$, where we consider $x+y$ and $x-y \in v_2(\mathbb{P}^4) = \Sigma_0$. $\square$

The projective dual $\Sigma_0^*$ is known to be the determinantal hypersurface in $(\mathbb{P}^{14})^* = \mathbb{P}(\mathrm{Sym}^2\check{\mathbb{C}}^5)$. Further, the duality reverses the inclusion $\Sigma \supset \Sigma_0$ to $\Sigma^* \subset \Sigma_0^*$ with $\Sigma^*$ being the singular locus of $\Sigma_0^*$ consisting of $5 \times 5$ matrices of rank $\leq 3$ [GKZ2, Chap.1, §.4.C]. Set $\mathcal{H} := (\Sigma_0)^*$ and consider the correspondence [Ty, Chap.3, §.3]:

$$\mathcal{U} = \{(x, A) \in \mathbb{P}^4 \times \mathcal{H} \mid Ax = 0\} \subset \mathbb{P}^4 \times \mathcal{H}.$$

Then $\mathcal{U}$ is a natural resolution of $\mathcal{H}$. To see the geometry of $\mathcal{U}$, denote the projection to the first and second factors by $\pi_1$ and $\pi_2$, respectively. Then the morphism $\pi_2 : \mathcal{U} \to \mathcal{H}$ is one to one over $\mathcal{H} \setminus \Sigma^*$ since $A$ is of rank 4. Over a point of rank $i$, the fiber is isomorphic to $\mathbb{P}^{4-i}$. The first projection $\pi_1$ to $\mathbb{P}^4$ represents $\mathcal{U}$ as a projective bundle over $\mathbb{P}^4$, where the fiber $\mathcal{U}_x$ over a point $x$ is the space of singular quadrics in $\mathbb{P}^4$ containing $x$ in their vertices. In particular, $\mathcal{U}$ is smooth. $\mathcal{U}_x$ can be also regarded as the space of quadrics in $\mathbb{P}^3$, where $\mathbb{P}^3$ is the image of the projection $\mathbb{P}^4 \dashrightarrow \mathbb{P}^3$ from $x$.

Let $\mathcal{M} := \pi_2^*\mathcal{O}_{\mathcal{H}}(1)$ and $\mathcal{L} := \pi_1^*\mathcal{O}_{\mathbb{P}^4}(1)$. Denote by $\mathcal{E}$ the $\pi_2$-exceptional divisor. The divisor $\mathcal{E}$ is contracted by $\pi_2$ to the locus of symmetric matrices of rank $\leq 3$.

**Proposition 3.5.** $\mathcal{E} = 4\mathcal{M} - 2\mathcal{L}$.

*Proof.* This is a specialization of [Ty, Chap. 3, §2, Lemma 3.1] and a generalization of [Co, Proposition 2.4.1]. We repeat the proof of [Ty, Chap. 3, §2, Lemma 3.1] for the readers' convenience.

Since $\mathcal{L}$ and $\mathcal{M}$ freely generate $\mathrm{Pic}\,\mathcal{U}$, we may write $\mathcal{E} = x\mathcal{M} - y\mathcal{L}$ for some integers $x$ and $y$. Recall that a fiber $F$ of $\pi_1 : \mathcal{U} \to \mathbb{P}^4$ is isomorphic to the space of quadrics in $\mathbb{P}^3$, and then $\mathcal{E}|_F$ is identified with the space of singular quadrics. Thus $\mathcal{E}|_F$ is the determinantal hypersurface of degree 4 in $F \simeq \mathbb{P}^9$. Restricting $\mathcal{E} = x\mathcal{M} - y\mathcal{L}$ to $F$ and noting $\mathcal{L}|_F = 0$ and $\mathcal{M}|_F = \mathcal{O}_{\mathbb{P}^9}(1)$, we have $x = 4$.

Let $Q \in \mathcal{H}$ be a quadric of rank three. Then the fiber $\gamma$ of $\pi_2 : \mathcal{U} \to \mathcal{H}$ over $Q$ is isomorphic to $\mathbb{P}^1$ and is mapped to the vertex of $Q$ by $\pi_1$. By the following lemma, we have $\mathcal{E} \cdot \gamma = -2$. Restricting $\mathcal{E} = x\mathcal{M} - y\mathcal{L}$ to $\gamma$ and noting $\mathcal{L}|_\gamma = \mathcal{O}_{\mathbb{P}^1}(1)$ and $\mathcal{M}|_\gamma = 0$, we have $y = 2$. $\square$

**Lemma 3.6.** *Let $Q \in \mathcal{H}$ be a quadric of rank three. Then, analytically locally near $Q$, $\mathcal{H}$ is isomorphic to the product of the 2-dimensional ordinary double point and $\mathbb{C}^{11}$.*

*Proof.* This is [Ty, Chap. 2, §3, Lemma 2.3]. The proof is similar to that of Lemma 3.2. In that proof, we have only to take as B a symmetric matrix. $\square$

**(3-4)** *Hessian quintic of the Reye congruence $X$*: Since the Chow embedding $\Sigma \hookrightarrow \mathbb{P}(\mathrm{Sym}^2\mathbb{C}^5)$ is defined by symmetric $(1,1)$-divisors, we have for $X \cong \tilde{X}/\langle\sigma\rangle$:

$$X = \Sigma \cap H_1 \cap \cdots \cap H_5 \subset \mathbb{P}(\mathrm{Sym}^2\mathbb{C}^5),$$

where $H_i$ are linear forms on $\mathbb{P}(\mathrm{Sym}^2\mathbb{C}^5)$ representing the quadratic forms $Q_i$ on $\check{\mathbb{P}}^4$. The Mukai dual of $X$ has dimension one. Since we expect a threefold related to $X$, we shift by one in the inclusion $\Sigma_0 \subset \Sigma$ and define the *shifted* Mukai dual by

$$H = (\Sigma_0)^* \cap \langle h_1, \cdots, h_5 \rangle \subset \mathbb{P}(\mathrm{Sym}^2\check{\mathbb{C}}^5),$$

where $(\Sigma_0)^* = \mathcal{H}$ is the determinantal hypersurface. In our context, this definition may be regarded as the symmetric limit of the Mukai dual $\tilde{X}_0^\sharp$.

By construction, the projective space $\mathbb{P}_h^4 := \langle h_1, \cdots, h_5 \rangle$ is nothing but the linear system $P$ of the quadrics. The shifted Mukai dual $H$ corresponds to *the Hessian surface* in the classical Reye congruence [Co]. Explicitly $H$ may be written by

$$H = \{ [y] \in \mathbb{P}^4 \mid \det(\sum y_i A_i) = 0 \},$$

where we recall that $A_i$ $(i = 1, \ldots, 5)$ are bases of $P$.

We may now construct a diagram similar to (3.1). Since $A_i$ are symmetric, we have the matrices $B_i$ satisfying

$$\sum y_i A_i = {}^t(B_1 y \ B_2 y \ B_3 y \ B_4 y \ B_5 y) = (B_1 y \ B_2 y \ B_3 y \ B_4 y \ B_5 y),$$

where $y = {}^t(y_1 \ y_2 \ \cdots \ y_5)$. Similarly to $\tilde{X}_1$ and $\tilde{X}_2$ in the diagram (3.1), we define

$$U_1 = \{ (y, w) \in H \times \mathbb{P}^4 \mid (\sum y_i A_i) w = 0 \} = \{ (y, w) \mid {}^t y \, {}^t B_i w = 0 \, (1 \le i \le 5) \},$$

$$U_2 = \{ (z, y) \in \mathbb{P}^4 \times H \mid {}^t z (\sum y_i A_i) = 0 \} = \{ (z, y) \mid {}^t z B_i y = 0 \, (1 \le i \le 5) \}.$$

Corresponding to $Z_1$ and $Z_2$ in (3.1), we define

$$S_1 = \{ w \in \mathbb{P}^4 \mid \det (A_1 w \cdots A_5 w) = 0 \}, \ S_2 = \{ z \in \mathbb{P}^4 \mid \det (A_1 z \cdots A_5 z) = 0 \}.$$

As before, we have natural maps $U_i \to H$, $U_i \to S_i$ and $\tilde{X} \to S_i$ $(i = 1, 2)$ and they are all birational and crepant.

We summarize our construction into a diagram:

$$(3.2)$$



In this diagram, we have included a Calabi-Yau manifold $Y$ which will be constructed in the next subsection. Since $A_i$ are symmetric matrices, obviously $S_1 \simeq S_2$ and $U_1 \simeq U_2$. We sometimes abbreviate $U_i$ and $S_i$ as $U$ and $S$ respectively.

We study further properties of the diagram (3.2) assuming only smoothness of $\tilde{X}$ to the end of this subsection.

**Claim 3.7.** $U$ is smooth.

*Proof.* $S$ is normal and, moreover, $S$ has only canonical singularities since $\tilde{X}$ is smooth and $\tilde{X} \to S$ is crepant. Since $U \to S$ is crepant, $U$ is normal and has only canonical singularities. Let $p \colon U' \to U$ be a crepant terminalization [Ko], i.e., $p$ is birational, $U'$ is $\mathbb{Q}$-factorial and has only terminal singularities, and $K_{U'} = p^* K_U$. Thus $U'$ and $\tilde{X}$ are two birational minimal models and then is isomorphic in codimension one. Since $\rho(\tilde{X}) = 2$, it holds that $\rho(U') = 2$. Since $U$ is a complete

intersection in $\mathbb{P}^4 \times \mathbb{P}^4$, $\rho(U)$ is at least two. Therefore $U' = U$. Since two birational minimal models have the same type of singularities [Ko], $U$ is also smooth. □

Let $L_0$ and $M_0$ be hyperplane sections on $S$ and $H$, respectively, and set $L = \pi_1^* L_0$, $M = \pi_2^* M_0$.

**Proposition 3.8.** $U \to H$ *is a crepant divisorial contraction contracting a divisor* $E$ *to a smooth curve* $C$ *of degree* 20 *and genus* 26. $H$ *has an* $A_1$ *type singularity along* $C$.

*Proof.* Since $\mathcal{U} \to \mathcal{H}$ contracts the divisor $\mathcal{E}$ to the locus $\Sigma^*$ of symmetric matrices of rank less than or equal to 3, which has codimension two in $\mathcal{H}$, $U \to H$ contracts the divisor $E := \mathcal{E}|_U$ to the curve $C := \Sigma^*|_H$. Since $U$ is smooth and $\rho(U) = 2$, $U \to H$ is a primitive contraction, hence, $E$ is irreducible and, by [Wi, Theorem 2.2], $C$ is a smooth curve. Moreover $C$ does not contain the point corresponding to a rank two quadric. Indeed, if $C$ contains the point corresponding to a rank two quadric $Q$, the fiber of $U \to H$ over $[Q]$ is isomorphic to $\mathbb{P}^2$ (the vertex of $Q$), and then $E$ is not irreducible, a contradiction. Hence any fiber of $E \to C$ is $\mathbb{P}^1$ since so is any fiber of $\mathcal{E} \to \Sigma^*$ over the point corresponding to a rank three quadric. Therefore $H$ has an $A_1$ type singularity along $C$. Since $\deg \Sigma^* = 20$ by the formula [HTu, Proposition 12] and $C$ is a linear section of $\Sigma^*$, $C$ has the same degree. To show $g(C) = 26$, we use the identity $2L = 4M - E$, which follows from Proposition 3.5. We have the following table of intersections:

- $M^3 = 5$ and $L^3 = 5$ since $H$ and $S$ are quintics.
- $M^2 E = 0$ since $M^2 E = (M_{|E})^2$ and $M_{|E}$ is numerically a sums of fibers of $E \to C$.
- $ME^2 = -2 \deg C = -40$. Indeed, note that $ME^2 = M_{|E} E_{|E}$. Since $M_{|E}$ is the pull-back of a hyperplane section of $C$ and $E_{|E} \cdot f = -2$ for a fiber $f$ of $E \to C$, we have $ME^2 = -2 \deg C$.
- $E^3 = (K_U + E)^2 E = (K_E)^2 = 8(1 - g(C))$ since $K_U = 0$.

Therefore we have

$$40 = (2L)^3 = (4M - E)^3 = 64 \times 5 - 12 \times 40 - E^3 = -160 - 8(1 - g(C)).$$

Hence $g(C) = 26$. □

**Claim 3.9.** *The type of* $\tilde{X} \to S$ *is one of the following:*

(1) $\tilde{X} \to S$ *contracts an irreducible divisor to a curve, or*
(2) $\tilde{X} \to S$ *contracts only a finite number of rational curves.*

*Proof.* Since $\rho(\tilde{X}) = 2$ and $\tilde{X}$ is smooth, $\tilde{X} \to S$ satisfies (1) or (2), or $\tilde{X} \to S$ contracts an irreducible divisor to a point. We exclude the last possibility. Suppose $\tilde{X} \to S$ contracts an irreducible divisor to a point. We denote by $E_i$ the exceptional divisor of $\tilde{X} \to S_i$ ($i = 1, 2$). In this case, $S$ is $\mathbb{Q}$-factorial, thus the image of $E_1$ on $S_2$ is an ample divisor since $\rho(S_2) = 1$. This implies that $E_1 \cap E_2 \neq \emptyset$ since $E_1$ is not ample on $\tilde{X}$. Then, however, the curve $E_1 \cap E_2$ is contracted by both $\tilde{X} \to S_1$ and $\tilde{X} \to S_2$, a contradiction since $\tilde{X} \to S_1$ and $\tilde{X} \to S_2$ are distinct and primitive. □

Assume that $\tilde{X} \to S$ contracts an irreducible divisor to a curve. We can construct examples for this situation. For example, the linear system $P$ determined by the

following symmetric matrices $A_i$ is regular and has the property:

$$\sum_{k=1}^{5} y_k A_k = \begin{pmatrix} y_2 & y_1 & 0 & 0 & y_5 \\ y_1 & y_3 & y_2 & 0 & 0 \\ 0 & y_2 & y_4 & y_3 & 0 \\ 0 & 0 & y_3 & y_5 & y_4 \\ y_5 & 0 & 0 & y_4 & y_1 \end{pmatrix}.$$

In this case, $S$ is $\mathbb{Q}$-factorial and the image of the divisor is $\operatorname{Sing} S$. Hence $U \to S$ cannot be a small contraction and then also contracts a divisor. The image of the exceptional divisor of $U \to S$ is also $\operatorname{Sing} S$. By [Wi, Theorem 2.2] and its proof, both $\tilde{X} \to S$ and $U \to S$ are the blow-ups along $\operatorname{Sing} S$, thus they are isomorphic over $S$. If we identifies $U_1$, $\tilde{X}$, and $U_2$, then $U_1 \to H$ is identified with $\tilde{X} \to S_2$. Therefore we can simplify the diagram (3.2) as follows:

(3.3)

$$
\begin{array}{ccccc}
Y & & & & Y \\
\downarrow{\scriptstyle \rho\ 2:1} & & \tilde{X} & & \downarrow{\scriptstyle \rho\ 2:1} \\
& \swarrow & \downarrow{\scriptstyle \pi\ 2:1} & \searrow & \\
H & & & & H \\
& & X. & &
\end{array}
$$

Assume that $\tilde{X} \to S$ contracts only finite number of rational curves. This is a general situation; if $P$ is general, then we can verify by computer calculations that $\tilde{X} \to S$ contracts 50 disjoint $\mathbb{P}^1$'s to 50 ordinary double points of $S$. In this case, $U \to S$ is also a small contraction since $S$ is not $\mathbb{Q}$-factorial. Moreover, $U$ and $\tilde{X}$ are not isomorphic since both of the Picard numbers are two and the types of contractions they have are different; $U$ has a divisorial and a small contraction but $\tilde{X}$ has two small contractions.

**(3-5)** *Construction of a double covering $Y$:* Set $N = 2M - L$ and $N_0 := \pi_{2*}N$. By Proposition 3.5, we have $2N \sim E$ whence $2N_0 \sim 0$.

**Proposition 3.10.** $N_0 \not\sim 0$.

*Proof.* By definition $M_0 = \pi_{2*}M$. Assume by contradiction that $N_0 \sim 0$, equivalently, $\pi_{2*}L \sim 2M_0$. Then $\pi_{2*}L$ is a Cartier divisor, hence we may write $L = \pi_2^*(\pi_{2*}L) - bE$ for some integer $b$. Therefore we have $N = \pi_2^*N_0 + bE \equiv bE$. Since $N \equiv \frac{1}{2}E$, we have $bE \equiv \frac{1}{2}E$, a contradiction. $\qquad\square$

Now we can take the double cover of $H$ associated to the 2-torsion Weil divisor $N_0$, namely,

(3.4)
$$Y := \mathcal{S}pec_H \left( \mathcal{O}_H \oplus \mathcal{O}_H(N_0) \right),$$

where $\mathcal{O}_H \oplus \mathcal{O}_H(N_0)$ has a ring structure by using a nowhere vanishing section of $2N_0 \sim 0$. The natural projection $\rho\colon Y \to H$ is ramified along the curve $C$ and is étale outside $C$. Since it is analytically locally a universal cover along $C$, we see that $Y$ is smooth. Set $\widetilde{M} = \rho^* M_0$.

**Proposition 3.11.** *The 3-fold $Y$ is a Calabi-Yau 3-fold with $\widetilde{M}^3 = 10$.*

*Proof.* By construction, we have $K_Y \sim 0$ and $\widetilde{M}^3 = 10$. We verify $h^i(\mathcal{O}_Y) = 0$ for $i = 1, 2$. Indeed, by $\rho_* \mathcal{O}_Y = \mathcal{O}_H \oplus \mathcal{O}_H(N_0)$, we have $h^i(\mathcal{O}_Y) = h^i(\mathcal{O}_H) + h^i(\mathcal{O}_H(N_0)) = h^i(\mathcal{O}_H(N_0))$. Thus it suffices to show $h^i(\mathcal{O}_H(N_0)) = 0$. Note that a similar construction to the above works for a general 5-dimensional linear system $\overline{P}$ of quadrics in $\mathbb{P}^4$. We choose such a $\overline{P}$ containing $P$. We obtain several objects corresponding to those for $P$. We denote them by putting overlines to the corresponding objects for $P$. First of all, note that $\overline{H}$ is a quintic hypersurface in $\mathbb{P}^5$, thus it is a Fano 4-fold. For a general $\overline{P}$, $\overline{H}$ does not contain the point corresponding to a rank two quadric and then $\overline{H}$ has only ordinary double points as its singularities. Moreover, $\overline{S} = \mathbb{P}^4$. By Proposition 3.5, we can show that $2\overline{L} = 4\overline{M} - \overline{E}$. We set $\overline{N}_0 := \overline{\pi_{2*}}(2\overline{M} - \overline{L})$. Then $\overline{N}_0$ is a 2-torsion Weil divisor on $\overline{H}$ and $\overline{N}_0|_H = N_0$. Consider the exact sequence

$$0 \to \mathcal{O}_{\overline{H}}(\overline{N}_0 - H) \to \mathcal{O}_{\overline{H}}(\overline{N}_0) \to \mathcal{O}_H(N_0) \to 0.$$

Since $\overline{H}$ is Fano, we have

$$h^i(\mathcal{O}_{\overline{H}}(\overline{N}_0)) = h^i(\mathcal{O}_{\overline{H}}(\overline{N}_0 - K_{\overline{H}} + K_{\overline{H}})) = 0 \, (i = 1, 2)$$

by the Kawamata-Viehweg vanishing theorem. Similarly,

$$h^{i+1}(\mathcal{O}_{\overline{H}}(\overline{N}_0 - H)) = h^{3-i}(\mathcal{O}_{\overline{H}}(-\overline{N}_0 + H + K_{\overline{H}})) = 0 \, (i = 1, 2).$$

Therefore we have $h^i(\mathcal{O}_H(N_0)) = 0 \, (i = 1, 2)$.                    $\square$

**Proposition 3.12.** $c_2(Y) \cdot \widetilde{M} = 40$, $e(Y) = -50$ *and* $\rho(Y) = 1$.

*Proof.* First let us compute $c_2(Y) \cdot \widetilde{M}$. Take a general member $D \in |M_0|$ and set $\widetilde{D} := \rho^* D$ and $\rho' = \rho|_{\widetilde{D}}$. By the Bertini theorem, $\widetilde{D}$ is smooth, and $D$ intersects $C$ transversely at 20 points, and $D$ has ordinary double points at $D \cap C$. By the standard exact sequence

$$0 \to T_{\widetilde{D}} \to T_Y|_{\widetilde{D}} \to \mathcal{O}_{\widetilde{D}}(\widetilde{D}) \to 0,$$

we have

$$c_2(Y) \cdot \widetilde{M} = c_2(T_Y|_{\widetilde{D}}) = c_1(\widetilde{D}) \cdot \widetilde{D}|_{\widetilde{D}} + c_2(\widetilde{D}) = c_2(\widetilde{D}) - 10.$$

By the Noether formula, we have $c_2(\widetilde{D}) = 12\chi(\mathcal{O}_{\widetilde{D}}) - c_1^2(\widetilde{D}) = 12\chi(\mathcal{O}_{\widetilde{D}}) - 10$. Since $\rho'$ is the restriction of the double cover $\rho$, we have $\rho'_* \mathcal{O}_{\widetilde{D}} = \mathcal{O}_D \oplus \mathcal{O}_D(n_0)$, where $n_0 = N_0|_D$. Thus $\chi(\mathcal{O}_{\widetilde{D}}) = \chi(\mathcal{O}_D) + \chi(\mathcal{O}_D(n_0))$. Since $D$ is a quintic surface, it holds $\chi(\mathcal{O}_D) = h^0(\mathcal{O}_D) + h^0(K_D) = 5$. To compute $\chi(\mathcal{O}_D(n_0))$, we use the singular Riemann-Roch theorem for surfaces with only Du Val singularities [Re, Theorem 9.1]. Then, noting that $2n_0 \sim 0$ and $n_0$ is not Cartier at 20 ordinary double points of $D$, we have $\chi(\mathcal{O}_D(n_0)) = \chi(\mathcal{O}_D) - \frac{1}{4} \times 20 = 0$. Consequently we have $c_2(Y) \cdot \widetilde{M} = \chi(\mathcal{O}_D) - 20 = 40$.

Second we compute $e(Y) = -50$. Recalling $\widetilde{X} \simeq U$ or $\widetilde{X}$ and $U$ are connected by a flop, we have $e(U) = e(\widetilde{X}) = -100$ (cf. [Ko]). Since $2N = E$, we can take the double covering $\widetilde{U} \to U$ branched along $E$. By $2e(U) = e(\widetilde{U}) + e(E)$, we have

$e(\widetilde{U}) = -100$. This double covering is compatible with $Y \to H$ as follows:

$$
\begin{array}{ccc}
\widetilde{U} & \xrightarrow{\widetilde{\pi_2}} & Y \\
\downarrow & & \downarrow{\scriptstyle \rho} \\
U & \xrightarrow{\pi_2} & H,
\end{array}
$$

where $\widetilde{\pi_2}$ is the blow-up along the smooth curve $\rho^{-1}C \simeq C$. Since $e(C) = -50$, we have $e(Y) = -50$.

Finally we show $\rho(Y) = 1$. Since $e(Y) = 2(h^{1,1}(Y) - h^{1,2}(Y)) = -50$, we have only to show $h^{1,2}(Y) = 26$. It is well-known that $h^{1,2}(Y)$ is the number of moduli of $Y$. First we compute the number of moduli of $H$ by the same way as [Ca, Proposition 2.26] as follows. If $H$ is isomorphic to another symmetric determinantal quintic $H'$ in $P$, then this isomorphism is induced by a projective automorphism of $P$ since $\rho(H) = \rho(H') = 1$ and hence the isomorphism preserve the primitive polarization. An automorphism $H \simeq H$ induces an automorphism $Y \simeq Y$. Since $Y$ is a smooth Calabi-Yau 3-fold, its automorphism group is finite, thus so is the automorphism group of $H$. Therefore the number of moduli of $H$ is $\dim \mathbb{P}(\mathrm{Hom}(\mathbb{C}^5, \mathrm{Sym}^2 \mathbb{C}^5))/\mathrm{Aut}\, P \times \mathrm{Aut}\, \mathbb{P}^4 = 26$. Next note that any deformation of $Y$ comes from that of the pair $(Y, \tilde{M})$ since $h^2(\mathcal{O}_Y) = 0$ by [Se, p.151, Proposition 3.3.12]. Hence a deformation of $Y$ induces that of $H$, and then the number of moduli of $Y$ is less than or equal to that of $H$. Therefore $h^{1,2}(Y) \leq 26$, equivalently, $h^{1,1}(Y) \leq 1$, which must be an equality.  □

**Remark 3.13.** The surface $D$ and $\widetilde{D}$ constructed in the proof of Proposition 3.12 is originally studied deeply by F. Catanese [Ca] (see also [Be]). Our $Y$ and $H$ are three-dimensional counterpart of his surface.

From Proposition 3.11 and Proposition 3.12, we conclude that

**Theorem 3.14.** *The double cover $Y = Spec_H(\mathcal{O}_H \oplus \mathcal{O}_H(N_0))$ of the Hessian quintic is the Calabi-Yau threefold predicted in Conjecture 1.*

## 4. Counting curves and discussions

**(4-1)** *Counting curves on $X$ and $Y$*: We verify some integer numbers called BPS numbers which we read from the higher genus Gromov-Witten invariants determined by mirror symmetry (see Appendix A). We observe that some numbers have good interpretations from the geometry of Calabi-Yau manifolds $X$ and $Y$.

(4-1.1) *Genus 0 curve of degree 1 on $X$, $n_0^X(1) = 50$*: If $\tilde{X} \neq U$, then there are exactly 50 exceptional curves $l_j{}^i$ of $\tilde{X} \to S_i$ in general ($i = 1, 2, j = 1, \ldots, 50$). We label them so that $l_j{}^1$ and $l_j{}^2$ are exchanged by the involution. Since $\deg l_j{}^i = 1$ with respect to the $(1, 1)$ divisor class on $\tilde{X}$, the image $n_j$ of $l_j{}^1$ and $l_j{}^2$ on $X$ have degree 1. Thus $n_j$ are lines on $X$ and are mutually distinct.

If $\tilde{X} = U$, then $S_i = H$ and non-trivial fibers of $\tilde{X} \to S_i$ are parametrized by $C$, whose Euler number is $-50$ (see Proposition 3.8). Similarly to the generic case, we can show that there is a family of lines on $X$ parametrized by $C$. According to the Gromov-Witten theory, we count the BPS number by $n_1^X(1) = (-1)^{\dim C} e(C) = 50$.

(4-1.2) *Genus 1 curve of degree 2 on Y, $n_1^Y(2) = 50$:* If $\tilde{X} \neq U$, then there are exactly 50 exceptional curves $\ell_j$ of $U \to S$ in general $(j = 1, \ldots, 50)$. Note that $\ell_j$ are the strict transforms of lines $m_j$ on $H$, thus $M \cdot \ell_j = 1$. By the identity $2L = 4M - E$ and $L \cdot \ell_j = 0$, $\ell_j$ intersects $E$ at four points counted with multiplicities.

(*) *For a general $P$, any $l_j$ intersects $E$ at four points transversally.*

*Proof.* This follows from a simple dimension count. We make use of the diagram $\mathbb{P}^4 \xleftarrow{\pi_1} \mathcal{U} \xrightarrow{\pi_2} \mathcal{H}$ defined in the subsection (3-3). Recall the notation there. We count the dimension of the locus $\mathcal{S}_{\mathcal{E}}$ of lines in fibers which are tangent to $\mathcal{E}$. A fiber $F$ of $\mathcal{U} \to \mathbb{P}^4$ is isomorphic to $\mathbb{P}^9$ and $\mathcal{E}|_F$ is a quartic hypersurface. Therefore, in $G(2, F)$, the locus of tangent lines to $\mathcal{E}|_F$ is 15-dimensional. Thus $\dim \mathcal{S}_{\mathcal{E}} = 15 + 4 = 19$. Let $l$ be a line in $(\mathbb{P}^{14})^*$. Then, in $G(4, (\mathbb{P}^{14})^*)$, the locus of 4-planes containing $l$ is 30-dimensional. Therefore, in $G(4, (\mathbb{P}^{14})^*)$, the locus of 4-planes containing the image of at least one tangent line to $\mathcal{E}$ is $(30 + 19)$-dimensional. On the other hand, $\dim G(4, (\mathbb{P}^{14})^*) = 50$. Hence a general $P \in G(4, (\mathbb{P}^{14})^*)$ does not contain the images of tangent lines to $\mathcal{E}$. $\square$

Due to the property (*), $m_j$ intersects with $C$ at four points transversally. Therefore by taking the double cover $Y \to H$, the inverse image of $m_j$ on $Y$ is an elliptic curve of degree 2.

If $\tilde{X} = U$, then we can similarly show that there is a family of elliptic curves on $Y$ of degree 2 parametrized by $C$. According Gromov-Witten theory again, we count the Euler number of $C$ for the BPS number $n_1^Y(2) = (-1)^{\dim C} e(C) = 50$.

(4-1.3) *Genus 7 curve of degree 8 on Y, $n_7^Y(8) = 150$:* We show that for each elliptic curve of degree two as in (4-1.2), there exists a family of genus 7 curves of degree 8 parametrized by $\mathbb{P}^2$. Indeed, an elliptic curve $\tilde{\delta}$ as in (4-1.2) is the inverse image of a 4-secant line $\delta$ of $C$. Let $\Pi$ be a plane in $P$ containing $\delta$. Note that such planes $\Pi$ are parametrized by a copy of $\mathbb{P}^2$. Then $\Pi \cap H$ is the union of $\delta$ and a curve $\gamma$ of degree 4 and (arithmetic) genus 3. For a generic $\Pi$, $\gamma$ is smooth, and $\gamma$ intersects the ramification curve $C$ of $\rho$ at the four points $C \cap \delta$. Moreover, the ramification $\tilde{\gamma} := \rho^{-1}(\gamma) \to \gamma$ occurs only at these four points and is simple. Hence $\deg \tilde{\gamma} = 8$ and, we can compute the genus of $\tilde{\gamma}$: $2g(\tilde{\gamma}) - 2 = 2(2g(\gamma) - 2) + 4$, i.e., $g(\tilde{\gamma}) = 7$.

Elliptic curves as in (4-1.2) are parametrized by 50 points or the curve $C$ with $e(C) = -50$. Therefore $n_7^Y(8) = 50 \times e(\mathbb{P}^2) = 150$.

(4-1.4) *Genus 6 curve of degree 10 on X, $n_6^X(10) = 5$:* We construct a family of curves of genus 6 and degree 10 which are parametrized by $(\mathbb{P}^4)^*$. Then we can explain the BPS number $n_6^X(10) = 5$ by $(-1)^{\dim(\mathbb{P}^4)^*} e((\mathbb{P}^4)^*) = 5$. Let $L \cong \mathbb{P}^3$ be any hyperplane in $\mathbb{P}^4$ and set $C_L := \mathrm{Sym}^2 L \cap X$. $C_L$ is a linear section of $\mathrm{Sym}^2 L$ since $X$ is a linear section of $\mathrm{Sym}^2 \mathbb{P}^4$. Since the degree of $L \times L$ is 20, the degree of $\mathrm{Sym}^2 L$ is 10. We see that $C_L$ is a curve. Indeed, otherwise $C_L$ contains a 2-dimensional component and its degree is less than or equal to 10 because $C_L$ is a linear section of $\mathrm{Sym}^2 L$. However, $\mathrm{Pic}\, X$ is generated by its hyperplane section, whose degree is 35, a contradiction. Thus $C_L$ is a curve, and then $C_L$ is a linear complete intersection in $\mathrm{Sym}^2 L$. Note that $K_{\mathrm{Sym}^2 L} = -4D$, where $D$ is a hyperplane section of $\mathrm{Sym}^2 L$. Thus $K_{C_L} = D|_{C_L}$, and then $\deg K_{C_L} = (D^6)_{\mathrm{Sym}^2 L} = 10$. This means that the arithmetic genus of $C_L$ is 6. We can prove that the curve $C_L$ is smooth for a generic $L$. To prove this, set $\widetilde{C}_L := (L \times$

$L) \cap \widetilde{X}$. Since the morphism $\widetilde{C}_L \to C_L$ is the quotient by the fixed point free involution, it suffices to show that $\widetilde{C}_L$ is smooth for a generic $L$. Note that $L \times L \subset \mathbb{P}^4 \times \mathbb{P}^4$ is the scheme of zeros of a section of the vector bundle $\mathcal{O}_{\mathbb{P}^4 \times \mathbb{P}^4}(1,0) \oplus \mathcal{O}_{\mathbb{P}^4 \times \mathbb{P}^4}(0,1)$. Therefore $\widetilde{C}_L$ is the scheme of zeros of a section of the vector bundle $\mathcal{E}_{\widetilde{X}} := \mathcal{O}_{\mathbb{P}^4 \times \mathbb{P}^4}(1,0)|_{\widetilde{X}} \oplus \mathcal{O}_{\mathbb{P}^4 \times \mathbb{P}^4}(0,1)|_{\widetilde{X}}$. Moreover we may choose a symmetric section defining $\widetilde{C}_L$. Since $\mathcal{E}_{\widetilde{X}}$ is generated by symmetric sections, a generic $\widetilde{C}_L$ is smooth by the Bertini theorem for vector bundles [Muk3].

Here we extract the relevant invariants from the Table 2 in Appendix A:

(4.1)

| $g$ | 0 | $\cdots$ | 4 | 5 | 6 |
|---|---|---|---|---|---|
| $n_g^X(10)$ | 80360393750 | $\cdots$ | 1713450 | 100 | 5 |

It is worth while remarking that we see similar curves in the Calabi-Yau threefolds given by linear sections of $Gr(2,7)$, $X_G := Gr(2,7)_{1^7}$, whose BPS numbers may be found in [HK] up to $g \leq 5$[2]. From the table we read the relevant part:

(4.2)

| $g$ | 0 | $\cdots$ | 6 | 7 | 8 |
|---|---|---|---|---|---|
| $n_g^{X_G}(14)$ | 26782042513523921505 | $\cdots$ | 123676 | 392 | 7 |

The BPS number $n_8^{X_G}(14)$ can be explained by a family of genus 8 and degree 14 curves parametrized by $(\mathbb{P}^6)^*$. This family follows from hyperplanes $L \cong \mathbb{P}^5$ in $\mathbb{P}^6$ as above and $C_L = \wedge^2 L \cap X_G$. For generic $L$, $C_L$ is a curve given by a transverse linear section of $G(2,6)$, which has genus 8, degree 14, and was studied in details in [Muk2]. Again, we explain $n_8^{X_G}(14) = 7$ by $(-1)^{\dim (\mathbb{P}^6)^*} e((\mathbb{P}^6)^*)$.

(4-1.5) *BPS numbers from nodal curves*: The BPS numbers with a fixed degree $d$ are indexed by geometric genus $h$ which ranges from zero to the arithmetic genus $g = g_d$, as typically shown in (4.1) and (4.2). This structure was introduced based on the intuitions from physics in [GV], and possible geometric explanations of this have been given in [KKV]. According to the latter formulation, the number $n_h(d)$ counts the Euler number (up to sign) of the locus on the moduli space where the curves have $g - h$ nodes. Based on this, it has been proposed for the BPS numbers of genus $g - 1$ that

(4.3)
$$n_{g-1}(d) = -(-1)^{\dim \mathcal{M}_d^{g_d}} \{ e(\mathcal{C}) + (2g - 2) e(\mathcal{M}_d^{g_d}) \},$$

where $\mathcal{M}_d^{g_d}$ is the parameter space of the curves of degree $d$ and the arithmetic genus $g = g_d$, and $\mathcal{C}$ is the universal curves over $\mathcal{M}_d^{g_d}$. Some of BPS numbers in our tables can be reproduced easily by (4.3).

(i) *Genus 5 curve of degree 10 on $X$, $n_5^X(10) = 100$*: In this case, the universal curve $\mathcal{C}$ has a natural fibration over $X$ with fiber $(\mathbb{P}^2)^*$. To see this fibration, we note that an element of the universal curve is given by $C_L = \mathrm{Sym}^2 L \cap X$. We fix a point $(z, w)$ on $X$. Then, since $X$ is smooth, $z \neq w$ and the condition $(z, w) \in \mathrm{Sym}^2 L$ imposes two linearly independent conditions on the choice of $L$ parametrized by $(\mathbb{P}^4)^*$. From this, we obtain the claimed fibration, and $e(\mathcal{C}) = 3 \times e(X) = -150$. Then we can evaluate the formula (4.3) as $n_5(10) = -(-150 + 10 \times 5) = 100$ verifying the BPS number $n_5^X(10)$ in (4.1).

The BPS number $n_7^{X_G}(14)$ in (4.2) can be verified exactly in the same way by applying the formula (4.3). In this case, an element of the universal curve is given

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[2]The data up to $g \leq 10$ is available at http://www.ms.u-tokyo.ac.jp/~hosono/GW/GrPf_html

by $C_L = \wedge^2 L \cap X_G$ parametrized by $(\mathbb{P}^6)^*$. Fixing a point $\xi \in X_G$ determines a line in $\mathbb{P}^6$ and entails two linearly independent conditions on the choice of $L$, and hence gives rise to a fibration over $X_G$ with fiber $(\mathbb{P}^4)^*$. We evaluate the formula (4.3) by $n_6(14) = -(5 \times e(X_G) + 14 \times 7)$ with $e(X_G) = -98$ verifying the BPS number $n_6^{X_G}(14)$ in (4.2).

(ii) *Genus* 11 *curve of degree* 10 *on Y*, $n_{11}^Y(10) = 10$: We construct a family of curves of genus 11 and degree 10 which are parametrized by $G(2, P)$. Then we can explain the BPS number $n_{11}^Y(10) = 10$ by $(-1)^{\dim G(2,P)} e(G(2, P)) = 10$. Let $\Pi \subset P$ be a plane in $P$. Then $C_\Pi := \Pi \cap H$ is a plane curve of degree 5 and arithmetic genus 6. Let $\widetilde{C}_\Pi := \rho^{-1}(C_\Pi)$. We show that $\widetilde{C}_\Pi$ is a genus 11 curve of degree 10 for a generic $\Pi$ (in general, the arithmetic genus of $\widetilde{C}_\Pi$ is 11). Indeed, for a generic $\Pi$, $C_\Pi$ is smooth and is disjoint from the ramification curve $C$ of $\rho$. Therefore $\widetilde{C}_\Pi \to C_\Pi$ is étale and its degree is 10. By the Hurwitz formula, we can compute the genus of $\widetilde{C}_\Pi$: $2g(\widetilde{C}_\Pi) - 2 = 2(2g(C_\Pi) - 2) = 20$, i.e., $g(\widetilde{C}_\Pi) = 11$.

(iii) *Genus* 10 *curve of degree* 10 *on Y*, $n_{10}^Y(10) = 100$: In a similar way to (4-1.5)(i), we interpret this number from $n_{11}^Y(10) = 10$. The universal curve $\mathcal{C}$ over $\mathcal{M}_{10}^{11}$ has a natural fibration over $Y$ with fiber $G(2, 4)$. Indeed, for a point $y \in Y$, planes in $P$ through $\rho(y)$ is parametrized by a copy of $G(2, 4)$. Thus we see that $e(\mathcal{C}) = e(G(2, 4))e(Y) = -300$. By (4.3), we have $n_{10}^Y(10) = 100$.

**(4-2)** *Discussions*: The similarity of the curves $C_L$ in their construction above is not accidental. In fact, we have $X = \mathrm{Sym}^2 \mathbb{P}^4 \cap H_1 \cap \cdots \cap H_5 \subset \mathbb{P}(\mathrm{Sym}\mathbb{C}^5)$ (see (3-4)), while for the linear sections of the Grassmannian $X_G = Gr(2, 7) \cap H_1 \cap \cdots \cap H_7 \subset \mathbb{P}(\wedge^2 \mathbb{C}^7)$. The Mukai dual $X_G^\sharp$ of $X_G$ is nothing but the Pfaffian Calabi-Yau threefold, and the derived equivalence between $X_G$ and $X_G^\sharp$ follows from the projective homological duality [Ku1] or from the explicit construction of the kernel of the equivalence [BCa]. Our Calabi-Yau threefold $Y$ has been defined as the branched covering of the Hessian quintic which we have called a shifted Mukai dual in (3-4). The necessity of the shifting is a well-known phenomenon in the projective homological duality [Ku2]. However the covering construction of $Y$ is a new ingredient that appeared in the projective geometry of the Reye congruence $X$. A general framework to incorporate this covering as well as the relevant projective duality seems to be required [HT].

## APPENDIX A.  BPS NUMBERS OF $X$ AND $Y$

For a Calabi-Yau manifold $\bullet$, the BPS numbers $\{n_g^\bullet(d)\}$ are read off from the Gromov-Witten invariants $\{N_g^\bullet(d)\}$ through the following formula proposed from the arguments in physics [GV]:

$$(A.1) \qquad \sum_{g \geq 0} N_g^\bullet(d)\lambda^{2g-2} = \sum_{k|d} \sum_{g \geq 0} n_g^\bullet(d/k)\frac{1}{k}\left(2\sin\frac{k\lambda}{2}\right)^{2g-2} .$$

| $d$ | $g = 0$ | $g = 1$ | $g = 2$ | $g = 3$ |
|---|---|---|---|---|
| 1 | 50 | 0 | 0 | 0 |
| 2 | 325 | 0 | 0 | 0 |
| 3 | 1475 | 275 | 0 | 0 |
| 4 | 15325 | 4400 | 0 | 0 |
| 5 | 148575 | 84866 | 0 | 0 |
| 6 | 1885575 | 1583175 | 4400 | 0 |
| 7 | 24310650 | 30888200 | 536200 | −550 |
| 8 | 348616525 | 604676675 | 29838375 | 14350 |
| 9 | 5158310775 | 12044071475 | 1207458375 | 11555950 |
| 10 | 80360393750 | 241743300988 | 42169242200 | 1184416575 |
| 11 | 1287049795175 | 4897366348600 | 1341159720325 | 78638706125 |
| 12 | 21247935013725 | 99853114108900 | 40099463511075 | 4132376105575 |
| 13 | 358438400398475 | 2048292673120975 | 1146369224007075 | 187040727391700 |
| 14 | 6171544153689825 | 42223845013663600 | 3168908261261600 | 7622009409916000 |
| 15 | 108035835968890075 | 874235620542355546 | 853281016802276675 | 287256247725860175 |

| $d$ | $g = 4$ | $g = 5$ | $g = 6$ | $g = 7$ |
|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| 8 | 0 | 0 | 0 | 0 |
| 9 | −100 | 0 | 0 | 0 |
| 10 | 1713450 | 100 | 5 | 0 |
| 11 | 722228850 | 579975 | −200 | 0 |
| 12 | 103406622475 | 399883675 | 2000 | 0 |
| 13 | 9444029474450 | 119293055775 | 143795975 | 4425 |
| 14 | 662595383138450 | 18997158932125 | 121118688975 | 18994975 |
| 15 | 38944123712509700 | 2069810853535000 | 35619476408010 | 114318975250 |
| 16 | 2013111720646080925 | 175344589016056100 | 6103559347336200 | 65977694328550 |
| 17 | 94357347971238632650 | 12406278936653253300 | 745756863552486250 | 17913359215581675 |
| 18 | 4093411060955729478975 | 766225906495401998275 | 71992613535494881825 | 3146508705290392100 |

| $d$ | $g = 8$ | $g = 9$ | $g = 10$ | $g = 11$ |
|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| 13 | 0 | 0 | 0 | 0 |
| 14 | 57450 | 0 | 0 | 0 |
| 15 | −411925 | 150 | 0 | 0 |
| 16 | 105464925850 | −14495125 | 0 | 0 |
| 17 | 126509159387650 | 97409049625 | −16452900 | 4125 |
| 18 | 54474919888610025 | 262131821055725 | 101459279850 | −16598650 |
| 19 | 1365432929771063235 0 | 177453757177942875 | 613991099856975 | 132790770150 |
| 20 | 2403664871323486315025 | 62687258689222376200 | 637194272508657750 | 1682171918115410 |

| $d$ | $g = 12$ | $g = 13$ | $g = 14$ |
|---|---|---|---|
| 1 | 0 | 0 | 0 |
| ⋮ | ⋮ | ⋮ | ⋮ |
| 17 | 0 | 0 | 0 |
| 18 | 4125 | 0 | 0 |
| 19 | −14185500 | 0 | 0 |
| 20 | 261281853600 | 8586100 | 500 |
| 21 | 5515988462714275 | 847491960225 | 111934650 |
| 22 | 11952342577834431775 | 21928222551486975 | 3692486074300 |
| 23 | 1041458283804936114 0700 | 64120595692604535100 | 105978121788664050 |
| 24 | 5202734853892057059278550 | 7168232015633641977 8925 | 400537217143155608900 |
| 25 | 1763658604119550663892185975 | 4444907533782965262458 8375 | 55928109290947210 0209725 |

Table 2. BPS numbers $n_g^X(d)$ of the Reye congruence $X$ up to $g = 14$.

| $d$ | $g = 0$ | $g = 1$ | $g = 2$ |
|---|---|---|---|
| 1 | 550 | 0 | 0 |
| 2 | 19150 | 50 | 0 |
| 3 | 1165550 | 8800 | 0 |
| 4 | 106612400 | 2205050 | 1475 |
| 5 | 12279982850 | 571891188 | 3421300 |
| 6 | 1623505897500 | 145348456975 | 2866211100 |
| 7 | 235773848446900 | 36407863802400 | 1629661401800 |
| 8 | 36677428272627500 | 9007535459995025 | 754167588991150 |
| 9 | 6013303354178962900 | 2206445978587108100 | 306780142847805350 |
| 10 | 1027574672771836629150 | 536379778481962721435 | 114213423234737792750 |
| 11 | 181575714674126153579750 | 129650593880351912045400 | 39864084273176413792000 |
| 12 | 32984176212221309289675600 | 31205903422813957619133225 | 13248719910964644190631550 |

| $d$ | $g = 3$ | $g = 4$ | $g = 5$ |
|---|---|---|---|
| 1 | 0 | 0 | 0 |
| 2 | 0 | 0 | 0 |
| 3 | 0 | 0 | 0 |
| 4 | 0 | 0 | 0 |
| 5 | 100 | 0 | 0 |
| 6 | 8187125 | 825 | 0 |
| 7 | 19888077100 | 37966000 | 8600 |
| 8 | 23146244434500 | 212732689425 | 320681675 |
| 9 | 18532051742762800 | 454873290963350 | 3593733857100 |
| 10 | 11781287359871427050 | 585882543518117475 | 12728758671277375 |
| 11 | 6399895657742481953000 | 551632827711119126550 | 2453838743683422735 |
| 12 | 3101222971082783163969650 | 419397397316501722086175 | 3246547603195887816770 |
| 13 | 1377826723507606278941823100 | 272882663054775849898040250 | 331837670127937703360792… |

| $d$ | $g = 6$ | $g = 7$ | $g = 8$ |
|---|---|---|---|
| 1 | 0 | 0 | 0 |
| ⋮ | ⋮ | ⋮ | ⋮ |
| 6 | 0 | 0 | 0 |
| 7 | 0 | 0 | 0 |
| 8 | $-10750$ | 150 | 0 |
| 9 | 4127030600 | $-2187050$ | 1650 |
| 10 | 94413759088805 | 85277244900 | $-143177275$ |
| 11 | 506498658506640250 | 3797470967882100 | 3585047803950 |
| 12 | 136847718370265137 | 2830344004381456 | 229236705583292025 |
| 13 | 242384692201553443617 | 101000317775360734233450 | 2180564163043456331350 |
| 14 | 320892955620961007144 | 22926089412569720228677 | 97487227044956347748027… |

| $d$ | $g = 9$ | $g = 10$ | $g = 11$ |
|---|---|---|---|
| 1 | 0 | 0 | 0 |
| ⋮ | ⋮ | ⋮ | ⋮ |
| 8 | 0 | 0 | 0 |
| 9 | 0 | 0 | 0 |
| 10 | 6875 | 100 | 10 |
| 11 | $-7611379550$ | 3258450 | $-6600$ |
| 12 | 319959943617050 | $-378820282275$ | 942134550 |
| 13 | 20098911841165263850 | 45765221584457950 | $-10410420724200$ |
| 14 | 226533008318367558727225 | 2456978448082146184175 | 8727482292438634825 |
| 15 | 1212262421921910091110786350 | 30987133343627355178162050 | 40213689626831596871305… |

| $d$ | $g = 12$ | $g = 13$ | $g = 14$ |
|---|---|---|---|
| 1 | 0 | 0 | 0 |
| ⋮ | ⋮ | ⋮ | ⋮ |
| 11 | 0 | 0 | 0 |
| 12 | 93900 | 750 | 0 |
| 13 | 140631287850 | $-156011000$ | $-87500$ |
| 14 | 4557518136088150 | 15789026415450 | $-75723835350$ |
| 15 | 2092760001092944591400 | 2911773889110540400 | 2180928121176200 |
| 16 | 85091316259943255956875200 | 617071664475129547125225 | 1599147427910004974425 |

Table 3. BPS numbers $n_g^Y(d)$ of the covering $Y$ up to $g = 14$.

## Appendix B.  BPS numbers of $\tilde{X}_0$

Gromov-Witten invariants $N_g^\bullet(\beta)$ of a Calabi-Yau manifold $\bullet$ are defined for $\beta \in H_2(\bullet, \mathbb{Z})$ in general. Corresponding BPS numbers are read by generalizing the relation (A.1). In the tables below, BPS numbers $n_g^{\tilde{X}_0}(i,j)$ are listed, where $(i,j) = (\beta.H_1, \beta.H_2)$ with the generators $H_1, H_2$ of $H^2(\tilde{X}_0, \mathbb{Z})$ from the each factor of $\mathbb{P}^4 \times \mathbb{P}^4$.

| $i \setminus j$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 50 | 0 | 0 | 0 | 0 | 0 |
| 1 | 50 | 650 | 1475 | 650 | 50 | 0 | 0 |
| 2 | 0 | 1475 | 29350 | 148525 | 250550 | 148525 | 29350 |
| 3 | 0 | 650 | 148525 | 3270050 | 24162125 | 75885200 | 110273275 |
| 4 | 0 | 50 | 250550 | 24162125 | 545403950 | 5048036025 | 22945154050 |
| 5 | 0 | 0 | 148525 | 75885200 | 5048036025 | 114678709000 | 1231494256550 |
| 6 | 0 | 0 | 29350 | 110273275 | 22945154050 | 1231494256550 | 27995704239850 |
| 7 | 0 | 0 | 1475 | 75885200 | 55531376500 | 7175800860250 | 334030085380350 |
| 8 | 0 | 0 | 0 | 24162125 | 74278763500 | 24352783493100 | 2329042266808650 |
| 9 | 0 | 0 | 0 | 3270050 | 55531376500 | 50034381769600 | 10084936612321850 |
| 10 | 0 | 0 | 0 | 148525 | 22945154050 | 63477362571125 | 28126794522576400 |
| 11 | 0 | 0 | 0 | 650 | 5048036025 | 50034381769600 | 51642034298930775 |
| 12 | 0 | 0 | 0 | 0 | 545403950 | 24352783493100 | 63157566038079800 |

Table 4. BPS numbers $n_g^{\tilde{X}_0}(i,j)$ for $g = 0$.

| $i \setminus j$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3 | 0 | 0 | 0 | 1475 | 29350 | 148525 | 250550 | 148525 |
| 4 | 0 | 0 | 0 | 29350 | 2669500 | 46911250 | 303610050 | 882636150 |
| 5 | 0 | 0 | 0 | 148525 | 46911250 | 2311178040 | 38756326500 | 298784327925 |
| 6 | 0 | 0 | 0 | 250550 | 303610050 | 38756326500 | 1477879258975 | 24724246516200 |
| 7 | 0 | 0 | 0 | 148525 | 882636150 | 298784327925 | 24724246516200 | 824125289385950 |
| 8 | 0 | 0 | 0 | 29350 | 1249719025 | 1207298050100 | 217335663077200 | 13948250904141600 |
| 9 | 0 | 0 | 0 | 1475 | 882636150 | 2731112702750 | 1103600201154950 | 135767820281303350 |
| 10 | 0 | 0 | 0 | 0 | 303610050 | 3573290410020 | 3417167213249325 | 817523002761866550 |
| 11 | 0 | 0 | 0 | 0 | 46911250 | 2731112702750 | 6658383337394000 | 3186381984770132650 |
| 12 | 0 | 0 | 0 | 0 | 2669500 | 1207298050100 | 8301844531611000 | 8273823575633968400 |

Table 5. BPS numbers $n_g^{\tilde{X}_0}(i,j)$ for $g = 1$,

| $i \setminus j$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 4 | 0 | 0 | 0 | 0 | 0 | 2500 | 36800 | 132150 |
| 5 | 0 | 0 | 0 | 0 | 2500 | 2238300 | 101188225 | 1276419800 |
| 6 | 0 | 0 | 0 | 0 | 36800 | 101188225 | 10885677450 | 321011805475 |
| 7 | 0 | 0 | 0 | 0 | 132150 | 1276419800 | 321011805475 | 19785981206800 |
| 8 | 0 | 0 | 0 | 0 | 191850 | 6764994000 | 4007017841650 | 497495418446900 |
| 9 | 0 | 0 | 0 | 0 | 132150 | 17585700200 | 25512018106050 | 6354579702758450 |
| 10 | 0 | 0 | 0 | 0 | 36800 | 24017901850 | 91091698085900 | 46325408551373425 |
| 11 | 0 | 0 | 0 | 0 | 2500 | 17585700200 | 192086807308450 | 206305163005291900 |
| 12 | 0 | 0 | 0 | 0 | 0 | 6764994000 | 245649059538250 | 585085137096464550 |

Table 6. BPS numbers $n_g^{\tilde{X}_0}(i,j)$ for $g = 2$,

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